DYNAMICAL TITS ALTERNATIVE FOR GROUPS OF ALMOST AUTOMORPHISMS OF TREES

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Abstract. We prove a dynamical variant of the Tits alternative for the group of almost automorphisms of a locally finite tree $\mathcal{T}$: a group of almost automorphisms of $\mathcal{T}$ either contains a nonabelian free group playing ping-pong on the boundary $\partial\mathcal{T}$, or the action of the group on $\partial\mathcal{T}$ preserves a probability measure. This generalises to all groups of tree almost automorphisms a result of S. Hurtado and E. Militon for Thompson's group V, with a hopefully simpler proof.

1. Context and contributions

The Tits alternative is a celebrated theorem by J. Tits [Tit72] that shows a sharp dichotomy for linear groups over a field of characteristic zero: either they are virtually solvable or they contain a nonabelian free group. A group G is said to satisfy the Tits alternative if for every subgroup H of G, H is virtually solvable or contains a nonabelian free group. This group property has been established for a great deal of countable groups (see the references in [dlH00, Section II B, Complement 42]), usually by applying the Klein ping-pong lemma to exhibit free subgroups.

There are also many countable groups known to fail this alternative, as do many groups of homeomorphisms of compact spaces. For instance, the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of the circle and the group of automorphisms $\mathrm{Aut}(\mathcal{T})$ of a regular tree $\mathcal{T}$ of degree ≥ 3: the former contains Thompson's group F of piecewise affine dyadic homeomorphisms of $[0,1]$, the latter contains the first Grigorchuk group, and these subgroups are not virtually solvable and do not contain free groups (see [CFP96] and [Gri80], respectively). Nevertheless, these two examples satisfy a dynamical variant of this condition which we formulate as follows.

Definition 1.1. *Let X be a compact topological space and G a group of homeomorphisms of X. We say that the action of G on X* satisfies the dynamical Tits alternative *if for every subgroup H of G one of the following holds.*

- *The action of H preserves a regular probability measure on X.*
- *There exists a* ping-pong pair *for the action of H, that is, there exist $g, h \in H$ and $U_1, U_2, V_1, V_2 \subset X$ disjoint open sets such that*

$$g(X \setminus U_1) \subseteq V_1 \quad \textit{and} \quad h(X \setminus U_2) \subseteq V_2. \tag{1.1}$$

This dynamical alternative is a property of a group action, not merely of a group. Nonetheless, if $g, h \in H$ belong to a ping-pong pair, the ping-pong lemma shows that g, h generate a nonabelian free group. Moreover, the conditions in Definition 1.1 exclude each other and it suffices to verify them on finitely generated $H \leq G$, see the beginning of the proof of Theorem A.

Remark. Previous work [MM23, HM19] involving this notion define the dynamical Tits alternative as a weaker condition, where every subgroup H is required to preserve a probability measure or to contain a nonabelian free group. We prefer our definition since this weaker notion is not a

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dichotomy, and moreover all known proofs of the alternative yield the stronger condition. For instance, whenever $G \curvearrowright X$ satisfies Definition 1.1, a subgroup $H \leq G$ preserves a probability measure on X if and only if every pair of elements of H preserve a common probability measure on X.

Examples. A first family of examples comes from one-dimensional dynamics: the action on S^1 of the group of homeomorphisms of S^1 satisfies the dynamical Tits alternative by a theorem of G. Margulis [Mar00]. A related example is the Higman-Thompson group V acting on the triadic Cantor set, which also satisfies the alternative by work of S. Hurtado and E. Militon [HM19, Theorem 1.3]. A generalization of both statements is given in [MM23, Theorem 1.3], where it is shown that for any compact $K \subset \mathbb{R}$, the defining action of the group of locally monotone homeomorphisms of K satisfies the alternative. It is notable that the proof in [MM23] finds sufficiently proximal elements on a group G that does not preserve a measure on K by running a random walk on G, whereas the arguments in [Mar00, HM19] are "deterministic". Groups acting by homeomorphisms on dendrites also satisfy the alternative by work of B. Duchesne and N. Monod [DM18, Theorem 1.6].

A second family of examples consists of groups acting on the boundary of Gromov-hyperbolic spaces: a first elementary instance of this class is the action of automorphism group of a locally finite tree $\mathcal{T}$ on its boundary $\partial\mathcal{T}$, as follows easily from the well-known dynamical classification of subgroups of $\mathrm{Aut}(\mathcal{T})$, see [Tit70]. More generally, if (M, d) is a Gromov-hyperbolic and proper metric space such that $\mathrm{Isom}(M, d)$ acts cocompactly on M, then the action of $\mathrm{Isom}(M, d)$ on the Gromov boundary ∂M satisfies the dynamical Tits alternative (see [CCMT15], and also [AS22, Theorem 1.10] for a probabilistic version).

This note is concerned with almost automorphism groups of locally finite trees, which is a large family of locally compact and totally disconnected groups that arise as follows: let $\mathcal{T}$ be a locally finite rooted tree and $\partial\mathcal{T}$ be its space of ends. The group of rooted tree automorphisms $\mathrm{Aut}_{\mathrm{r}}(\mathcal{T})$ acts on $\partial\mathcal{T}$ preserving the so-called visual metric. The group $\mathrm{AAut}(\mathcal{T})$ of almost automorphisms of $\partial\mathcal{T}$ consists of all homeomorphisms of $\partial\mathcal{T}$ that are local homothecies for this metric, that is, that locally rescale it. The natural group topology on $\mathrm{AAut}(\mathcal{T})$ is not the compact-open topology, but the unique group topology such that $\mathrm{Aut}(\mathcal{T})$ is a compact open subgroup of $\mathrm{AAut}(\mathcal{T})$. See Section 2 for more precise statements.

If $d \geq 2$, $k \geq 1$ and $\mathcal{T}_{d,k}$ is the rooted tree where the root has k children and all other vertices have d children, then $\mathrm{AAut}(\mathcal{T}_{d,k})$ is known as a Neretin group. These groups are known to be simple [Kap99], compactly presented [LB17] and to contain the Higman-Thompson group $V_{d,k}$ as a dense subgroup. They are the first examples of compactly generated simple groups without lattices [BCGM12] and moreover admit no invariant random subgroups by a result of T. Zheng [Zhe19].

We show that the action of $\mathrm{AAut}(\mathcal{T})$ on $\partial\mathcal{T}$ satisfies the dynamical Tits alternative for any locally finite rooted tree $\mathcal{T}$.

Theorem A. *Let $\mathcal{T}$ be a locally finite rooted tree. The action of* $\mathrm{AAut}(\mathcal{T})$ *on the boundary $\partial\mathcal{T}$ satisfies the dynamical Tits alternative.*

Some interesting groups to which Theorem A applies are the groups V_G considered by V. Nekrashevych in [Nek18], where $G \leq \mathrm{Aut}_{\mathrm{r}}(\mathcal{T}_{2,2})$ is a self-similar group. Here V_G is the subgroup of $\mathrm{AAut}(\mathcal{T}_{2,2})$ generated by G and Higman-Thompson's V.

Fix a linear order on $\partial\mathcal{T}$ that is compatible with the tree structure. We denote by $V_{\mathcal{T}}$ the group of elements of $\mathrm{AAut}(\mathcal{T})$ acting in a locally order-preserving manner. For instance, $V_{\mathcal{T}_{d,k}}$ is the Higman-Thompson group associated to $\mathcal{T}_{d,k}$, and all topological full groups of irreducible infinite one-sided shifts of finite type are naturally subgroups of some $V_{\mathcal{T}}$, see [Led20, Theorem 3.8]. We call $V_{\mathcal{T}}$ the *Higman-Thompson group associated to* $\mathcal{T}$, although this name is not standard. The proof of Theorem B below gives a shorter and hopefully more conceptual approach to [HM19, Theorem 1.3] when specialized to the Higman-Thompson group V.

Theorem B. *Let H be a finitely generated subgroup of $V_{\mathcal{T}}$. Then the action of H on $\partial\mathcal{T}$ has a finite orbit or admits a ping-pong pair.*

We emphasize that Theorem B is not new, since the proof of [HM19, Theorem 1.3] shows that V verifies the (slightly stronger) conclusion of Theorem B and general arguments allow to extend the result from V to any group $V_{\mathcal{T}}$.

Two important ingredients in both proofs are a characterization of relatively compact subgroups of $\mathrm{AAut}(\mathcal{T})$ by A. Le Boudec and P. Wesolek [LBW19] and a description of the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$ by G. Goffer and W. Lederle [GL21] (building on work of O. Salazar-Díaz [SD10]).

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2. Preliminaries

We give some background on $\mathrm{AAut}(\mathcal{T})$, describe the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$, recall the definition of the Vietoris topology on closed subsets of a space and fix some notation. For more details on this material, see [GL21, LBW19, GL18].

Notation. Given a metric space (X, d), $A \subseteq X$ and $\varepsilon > 0$, we denote $A^{\varepsilon} = \{x \in X \mid d(x, A) \leq \varepsilon\}$. When $\mathcal{T}$ is a rooted tree, we write $\mathrm{Aut_r}(\mathcal{T})$ for the group of tree automorphisms of $\mathcal{T}$ that fix the root.

Almost automorphism groups of trees. Let $\mathcal{T}$ be a locally finite rooted tree with no leaves. We denote its root by v_0, and assume that all edges are directed away from v_0. A *caret* is a subtree of $\mathcal{T}$ consisting of a vertex, its children, and the edges between them. A subtree is *complete* if it is a union of carets, and when $\mathcal{T}_1, \mathcal{T}_2$ are rooted complete subtrees of $\mathcal{T}$ we denote by $\mathcal{T}_2 \setminus \mathcal{T}_1$ the union of all carets in $\mathcal{T}_2$ that are not included in $\mathcal{T}_1$. The set of leaves of a tree $\mathcal{T}$ is denoted by $\mathcal{L}\mathcal{T}$.

Let $\partial\mathcal{T}$ be the *space of ends* of $\mathcal{T}$, that is, the set of (one-sided) infinite directed paths starting at v_0. We equip $\partial\mathcal{T}$ with the topology induced by the *visual metric* defined as $d(\xi, \xi') = 2^{-N(\xi,\xi')}$ for $\xi, \xi' \in \partial\mathcal{T}$ where $N(\xi, \xi') \in \mathbb{N}$ is the smallest integer n such that $\xi_n \neq \xi'_n$. The space $(\partial\mathcal{T}, d)$ is totally disconnected and compact, and its topology has a basis of clopen balls $\partial\mathcal{T}_v = \{\xi \in \partial\mathcal{T} \mid v \text{ is in } \xi\}$ where $v \in V(\mathcal{T})$.

An *almost automorphism* of $\mathcal{T}$ is a homeomorphism g of $\partial\mathcal{T}$ such that there exists a partition of $\partial\mathcal{T}$ into clopen balls $D_1, \ldots, D_n$ and positive numbers $\lambda_1, \ldots, \lambda_n$ such that $d(g(y), g(z)) = \lambda_j d(y, z)$ for all $y, z \in D_j$. Such a partition is said to be *admissible* for g. Another way of viewing an almost automorphism is the following: take $\mathcal{T}_1, \mathcal{T}_2$ finite subtrees of $\mathcal{T}$ with root v_0, so $\mathcal{T} \setminus \mathcal{T}_1$, $\mathcal{T} \setminus \mathcal{T}_2$ are naturally rooted forests. Then any isomorphism $\overline{g}\colon \mathcal{T} \setminus \mathcal{T}_1 \to \mathcal{T} \setminus \mathcal{T}_2$ of rooted forests determines a $g \in \mathrm{AAut}(\mathcal{T})$, and conversely any almost automorphism arises in this manner, although not uniquely so.

Call an almost automorphism $g \in \mathrm{AAut}(\mathcal{T})$ *elliptic* if there exists a partition $\mathcal{P}$ of $\partial\mathcal{T}$ into clopen balls that is admissible for g and g-invariant, that is, such that $g\mathcal{P} = \mathcal{P}$. The following theorem is stated for Neretin groups in [LBW19], but the proof for any locally finite tree $\mathcal{T}$ is the same word by word.

Proposition 2.1 ([LBW19, Corollary 3.6])**.** *Let $H \leq \mathrm{AAut}(\mathcal{T})$ be finitely generated. The following are equivalent.*

- *H is relatively compact in $\mathrm{AAut}(\mathcal{T})$.*
- *Every element of H is elliptic.*
- *There is a partition $\mathcal{P}$ of $\partial\mathcal{T}$ into clopen balls such that $\mathcal{P}$ is admissible for every $h \in H$.*

Moreover, if these conditions hold and $\mathcal{Q}$ is any partition of $\mathcal{P}$ into clopen balls such that $\mathcal{Q}$ is admissible for every element of some generating set of H, then $\mathcal{P}$ can be chosen to be finer than $\mathcal{Q}$.

Fix a family of total linear orders $\{<_v\}_{v\in V(\mathcal{T})}$ indexed by the vertices of $\mathcal{T}$, where $<_v$ is an order on the children of $v \in V(\mathcal{T})$. This family induces a linear order $<$ on $\partial\mathcal{T}$ by declaring $\xi < \xi'$ if $\xi, \xi' \in \partial\mathcal{T}$ and $\xi_n <_{\xi_{n-1}} \xi'_n$, where $n = N(\xi, \xi') \in \mathbb{N}$. We call $g \in \mathrm{AAut}(\mathcal{T})$ a *Higman-Thompson element* if there exists an admissible partition $\mathcal{P}$ for g such that $g\big|_D$ is order-preserving for all $D \in \mathcal{P}$. The subset of Higman-Thompson elements of $\mathrm{AAut}(\mathcal{T})$ is a group that we denote by $V_\mathcal{T}$. We omit $\{<_v\}_{v\in V(\mathcal{T})}$ from the definition of $V_\mathcal{T}$ to keep the notation uncluttered, but for non-regular trees $\mathcal{T}$ the group $V_\mathcal{T}$ may depend on the choice of the orders $\{<_v\}_{v\in V(\mathcal{T})}$.

Since the elliptic elements of $V_\mathcal{T}$ are exactly the elements of finite order in $V_\mathcal{T}$, as in [LBW19] Proposition 2.1 yields the following result.

Corollary 2.2 ([LBW19, Corollary 3.7])**.** *Any finitely generated subgroup of $V_\mathcal{T}$ composed entirely of elliptic elements is finite.*

Dynamics of almost automorphisms. We will make use of a description of the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$, which is one of the subjects of [GL21]. Again, the proofs are given for $\mathcal{T} = \mathcal{T}_{d,k}$ but a careful reading shows that all arguments hold in the general case.

We need some definitions, following [GL21]: a *tree pair* is a tuple $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$ are finite complete trees with root v_0 and $\kappa\colon \mathcal{L}\mathcal{T}_1 \to \mathcal{L}\mathcal{T}_2$ is a bijection between the leaves of $\mathcal{T}_1$ and the leaves of $\mathcal{T}_2$. We say that a tree pair $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is *associated* to an element $g \in \mathrm{AAut}(\mathcal{T})$ if g arises from an isomorphism of rooted forests $\overline{g}\colon \mathcal{T}\setminus\mathcal{T}_1 \to \mathcal{T}\setminus\mathcal{T}_2$ such that $\kappa = \overline{g}\big|_{\mathcal{L}\mathcal{T}_1}$. In this case $\{\partial\mathcal{T}_v\}_{v\in\mathcal{L}\mathcal{T}_1}$ is an admissible partition for g. We will only consider tree pairs that are associated to some almost automorphism of $\mathcal{T}$. All tree pairs are associated to almost automorphisms when $\mathcal{T} = \mathcal{T}_{d,k}$ but this is not true in general since the connected components of $\mathcal{T}\setminus\mathcal{T}_1$ and $\mathcal{T}\setminus\mathcal{T}_2$ need not be isomorphic.

Consider an orbit $\mathcal{O} = \{u_0, \ldots, u_n\} \subseteq \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ of the partial action of κ on $\mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$, that is, $\mathcal{O}$ is such that $u_0, \ldots, u_{n-1} \in \mathcal{L}\mathcal{T}_1$, $u_1, \ldots, u_n \in \mathcal{L}\mathcal{T}_2$, $\kappa(u_j) = u_{j+1}$ for all $j = 0, \ldots, n-1$ and either

- $u_0 \notin \mathcal{L}\mathcal{T}_2$ and $u_n \notin \mathcal{L}\mathcal{T}_1$, or
- $u_0 \in \mathcal{L}\mathcal{T}_2$, $u_n \in \mathcal{L}\mathcal{T}_1$ and $\kappa(u_n) = u_0$.

The orbit $\mathcal{O}$ is said to be

- an *attracting chain* if u_n is a descendant of u_0 in $\mathcal{T}$, in which case u_n is called the *attractor* of the orbit,
- a *repelling chain* if u_0 is a descendant of u_n in $\mathcal{T}$, in which case u_0 is called the *repeller* of the orbit,
- a *periodic chain* if $\kappa(u_n) = u_0$, and
- a *wandering chain* if $u_0 \notin \mathcal{T}_2$ (that is, u_0 is a descendant of some leaf in $\mathcal{L}\mathcal{T}_2$) and $u_n \notin \mathcal{T}_1$ (that is, u_n is a descendant of some leaf in $\mathcal{L}\mathcal{T}_1$).

These options are mutually exclusive. We say that $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is a *revealing tree pair* if each connected component of $\mathcal{T}_1 \backslash \mathcal{T}_2$ contains a repeller and each connected component of $\mathcal{T}_2 \backslash \mathcal{T}_1$ contains an attractor. In this case, if $C \subseteq \mathcal{T}_1 \setminus \mathcal{T}_2$ is a connected component containing a repeller u_0, the orbit $\{u_0, \ldots, u_n\}$ is such that u_n is the root of C and u_0 is the unique repeller in C. Similarly, if $C \subseteq \mathcal{T}_2 \setminus \mathcal{T}_1$ is a connected component containing an attractor u_n, the orbit $\{u_0, \ldots, u_n\}$ is such that u_0 is the root of C and u_n is the unique attractor in C. Revealing tree pairs were introduced by O. Salazar-Díaz in [SD10] to describe the dynamics of individual elements of Thompson's V.

If $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is a revealing tree pair, then every orbit $\mathcal{O}$ is an attracting, repelling, periodic or wandering chain: assume that $\mathcal{O}$ is not periodic, so it must end in an element $u_n \in \mathcal{L}\mathcal{T}_2 \setminus \mathcal{L}\mathcal{T}_1$ and begin in an element $u_0 \in \mathcal{L}\mathcal{T}_1 \setminus \mathcal{L}\mathcal{T}_2$. If $u_0 \in \mathcal{T}_2$, then u_0 is the root of its component of $\mathcal{T}_2 \setminus \mathcal{T}_1$ and u_n must be the attractor in this component. If $u_n \in \mathcal{T}_1$, then u_n is the root of its component in $\mathcal{T}_1 \setminus \mathcal{T}_2$ and u_0 must be the repeller in this component. If neither happen, then $\mathcal{O}$ is wandering.

Theorem 2.3 ([GL21, Lemma 2.17]). *Every element of* $\operatorname{AAut}(\mathcal{T})$ *is associated to some revealing tree pair.*

As a consequence we deduce the following corollary, the proof of which uses ideas present in [GL21, Section 3.1] (compare [HM19, Lemma 5.5] for Higman-Thompson's group V).

Corollary 2.4. *If* $g \in \operatorname{AAut}(\mathcal{T})$ *there is a partition* $\partial\mathcal{T} = U_g \sqcup V_g$ *into* g*-invariant clopen subsets such that* U_g *is equal to the open subset of* $\xi \in \partial\mathcal{T}$ *that admit a neighborhood* $U \subseteq \partial\mathcal{T}$ *such that some positive power of* $g\big|_U$ *is an isometry onto* U*. Moreover, the following properties are verified.*

- *There is a positive power of* $g\big|_{U_g}$ *that is an isometry (for the visual metric of* $\partial\mathcal{T}$*).*
- *There are finitely many* g*-periodic points in* V_g*, which we denote* $\operatorname{Per}_{\mathrm{hyp}}(g)$*.*
- *There is a partition* $\operatorname{Per}_{\mathrm{hyp}}(g) = \operatorname{Per}_{\mathrm{rep}}(g) \sqcup \operatorname{Per}_{\mathrm{att}}(g)$ *into* repelling *and* attracting *periodic points, such that for every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *so that for all* $k \geq N$ *we have*

$$g^k(V_g \setminus \operatorname{Per}_{\mathrm{rep}}(g)^{\varepsilon}) \subseteq \operatorname{Per}_{\mathrm{att}}(g)^{\varepsilon} \quad \textit{and} \quad g^{-k}(V_g \setminus \operatorname{Per}_{\mathrm{att}}(g)^{\varepsilon}) \subseteq \operatorname{Per}_{\mathrm{rep}}(g)^{\varepsilon}. \tag{2.1}$$

Proof. Let $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ be a revealing pair associated to g. Write $\widetilde{U}_g = \bigsqcup \partial\mathcal{T}_u$ where the union is over all vertices $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ that are in periodic chains, and set $\widetilde{V}_g = \partial\mathcal{T} \setminus \widetilde{U}_g$. The sets $\widetilde{U}_g$ and $\widetilde{V}_g$ are g-invariant and clopen, and if $\partial\mathcal{T}_u \subseteq \widetilde{U}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$, there exists $n \in \mathbb{N}$ such that $g^n(\partial\mathcal{T}_u) = \partial\mathcal{T}_u$ and $g^n\big|_{\partial\mathcal{T}_u}$ is an isometry. By taking appropriate powers of g we see that there is a $k \in \mathbb{N}$ such that $g^k\big|_{\widetilde{U}_g}$ is an isometry.

Now let $\partial\mathcal{T}_u \subseteq \widetilde{V}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in an attracting chain $\{u_0, \ldots, u_n\}$. Then $g^n(\partial\mathcal{T}_{u_0}) \subsetneq \partial\mathcal{T}_{u_0}$ and $g^n\big|_{\partial\mathcal{T}_{u_0}}$ is a contraction, so in particular $g^n(\partial\mathcal{T}_u) \subsetneq \partial\mathcal{T}_u$ and $g^n\big|_{\partial\mathcal{T}_u}$ is a contraction. Thus there exists a unique g^n-fixed point $\xi_u \in \partial\mathcal{T}_u$, and it also verifies $\bigcap_{k\in\mathbb{N}} g^{kn}(\partial\mathcal{T}_u) = \{\xi_u\}$. Set

$$\operatorname{Per}_{\mathrm{att}}(g) = \{\xi_u \mid u \text{ is in an attracting chain}\}.$$

If $\partial\mathcal{T}_u \subseteq \widetilde{V}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in a repelling chain instead, the same argument applied to g^{-1} shows that for some $n \in \mathbb{N}$ there exists a unique g^n-fixed point $\xi_u \in \partial\mathcal{T}_u$ such that $\bigcap_{k\in\mathbb{N}} g^{-kn}(\partial\mathcal{T}_u) = \{\xi_u\}$. Set

$$\operatorname{Per}_{\mathrm{rep}}(g) = \{\xi_u \mid u \text{ is in a repelling chain}\}.$$

The sets $\operatorname{Per}_{\mathrm{att}}(g)$ and $\operatorname{Per}_{\mathrm{rep}}(g)$ are disjoint and finite. Moreover, their union gives all g-periodic points in $\widetilde{V}_g$ since any $\xi \in \partial\mathcal{T}_u$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in a wandering chain cannot be a g-periodic point: indeed, if the orbit $\{u_0, \ldots, u_n\}$ is wandering, then the connected component of $u_n \in \mathcal{L}\mathcal{T}_2 \backslash \mathcal{T}_1$ in $\mathcal{T}_2 \setminus \mathcal{T}_1$ has a root $u_{\mathrm{a}} \in \mathcal{L}\mathcal{T}_1$ which must be the first element of an attracting orbit. Thus

$$d\left(g^{n-k+j}\left(\partial\mathcal{T}_{u_k}\right), g^j\left(\xi_{u_{\mathrm{a}}}\right)\right) \xrightarrow{j\to\infty} 0$$

for every $k = 0, \dots, n$. In the same way, the root $u_{\mathrm{r}} \in \mathcal{L}\mathcal{T}_2$ of the connected component of $u_0 \in \mathcal{L}\mathcal{T}_1 \setminus \mathcal{T}_2$ in $\mathcal{T}_1 \setminus \mathcal{T}_2$ is the last element of a repelling orbit, so

$$d\left(g^{-k-j}\left(\partial \mathcal{T}_{u_k}\right), g^{-j}\left(\xi_{u_{\mathrm{r}}}\right)\right) \xrightarrow{j\to\infty} 0$$

for every $k = 0, \dots, n$. Since $\xi_{u_{\mathrm{r}}} \neq \xi_{u_{\mathrm{a}}}$, there are no periodic points in any $\partial \mathcal{T}_{u_k}$ for $k = 0, \dots, n$.

To prove that $\widetilde{V}_g$, $\mathrm{Per}_{\mathrm{att}}(g)$, $\mathrm{Per}_{\mathrm{rep}}(g)$ verify the last item of the corollary, by symmetry it suffices to prove the first statement in (2.1). To do this, take a clopen $U \subsetneq \widetilde{V}_g \setminus \mathrm{Per}_{\mathrm{rep}}(g)$ and $\varepsilon > 0$, and let $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$. If u is in an attracting or wandering chain, it is clear that there is an $N \in \mathbb{N}$ such that $g^k(U \cap \partial \mathcal{T}_u) \subseteq \mathrm{Per}_{\mathrm{att}}(g)^{\varepsilon}$ for all $k \geq N$. If u is in a repelling orbit $\{u_0, \dots, u_n\}$, then the equality

$$\bigcap_{k\in\mathbb{N}} g^{-kn}(\partial \mathcal{T}_u) \cap (U \cap \partial \mathcal{T}_u) = \varnothing$$

shows that there is a $l \in \mathbb{N}$ such that $g^{-kn}(\partial \mathcal{T}_u) \cap (U \cap \partial \mathcal{T}_u) = \varnothing$ for all $k \geq l$. The sets $g^r(\partial \mathcal{T}_u)$, $r = 0, \dots, n-1$ are pairwise disjoint, and we conclude that $\partial \mathcal{T}_u \cap g^k(U \cap \partial \mathcal{T}_u) = \varnothing$ for all $k \geq nl$. Upon replacing U by $g^r(U)$ for some sufficiently big $r \in \mathbb{N}$ we may assume that $U \cap \partial \mathcal{T}_u$ is empty for all $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ in a repelling orbit. Hence (2.1) holds for a sufficiently large $N \in \mathbb{N}$.

Up to now, $\widetilde{U}_g$, $\widetilde{V}_g$ satisfy all the required properties except that $\widetilde{V}_g$ could still contain some points that admit a neighborhood $U \subseteq \partial \mathcal{T}$ such that some positive power of $g\big|_U$ is an isometry onto U. Call the set of such points $\mathcal{I} \subseteq \widetilde{V}_g$. All elements in $\widetilde{V}_g \setminus (\mathrm{Per}_{\mathrm{rep}}(g) \sqcup \mathrm{Per}_{\mathrm{att}}(g))$ admit a neighborhood U such that $g^j(U) \cap U = \varnothing$ for all $j \neq 0$, so $\mathcal{I} \subseteq \mathrm{Per}_{\mathrm{rep}}(g) \sqcup \mathrm{Per}_{\mathrm{att}}(g)$ is finite. Thus setting $U_g = \widetilde{U}_g \sqcup \mathcal{I}$, $V_g = \widetilde{V}_g \setminus \mathcal{I}$ ensures that all properties in the statement of the corollary are verified. $\square$

In contrast with the trees $\mathcal{T}_{d,k}$, the boundary of an arbitrary locally finite tree $\mathcal{T}$ may contain isolated points and the three items in the previous corollary do not specify U_g, V_g uniquely. The definition of U_g in the statement of the corollary gives a canonical definition in any case.

Vietoris topology. Given a topological space X, we denote by 2^X the space of closed subsets of X equipped with the *Vietoris topology*, defined as the topology with subbasis

$$\{K \in 2^X \mid K \cap U \neq \varnothing,\, K \cap V = \varnothing\}$$

where U, V range over all open subsets of X. If X is compact metrizable then 2^X is also compact metrizable [Eng89, Section 4.5], and in this case any action by homeomorphisms of a countable group on X induces naturally an action by homeomorphisms on 2^X [Eng89, Section 3.12].

3. Proofs

We first gather some useful lemmas.

Lemma 3.1 (Neumann's lemma, [Neu54, Lemma 4.1])**.** *Let G be a group acting on a set X and assume that the action has no finite orbits. Then for every pair of finite subsets A, B of X there exists an element $g \in G$ such that $g(A) \cap B$ is empty.*

Lemma 3.2 ([MM23, Proposition 1.16])**.** *Let G be a group acting by homeomorphisms on a compact metric space (X, d). Assume that*

i. there exists a positive integer p such that for any $\varepsilon > 0$ there exist nonempty finite sets $A, B \subset X$ of cardinality at most p with $g(X \setminus A^{\varepsilon}) \subseteq B^{\varepsilon}$ for some $g \in G$, and

ii. there are no finite G-orbits.

Then the action of G on X has a ping-pong pair.

Proof. Notice that condition (i) is equivalent to the existence of finite sets A, B of X that work for *any* $\varepsilon > 0$: a pair (A, B) of nonempty subsets of X of cardinality at most p is called a *contraction pair* if for every neighborhood U, V of A, B respectively there exists $g \in G$ such that $g(X \setminus U) \subset V$. Then condition (i) and the compactness of X imply that there exists a contraction pair.

Moreover, if (A, B) is a contraction pair and $u, v \in G$ then $(u(A), v(B))$ is a contraction pair too. Thus, by (ii) and Lemma 3.1, our contraction pair (A, B) can be taken such that A and B are disjoint.

Using (ii) and Lemma 3.1 again we deduce that there exist two contraction pairs $(A_1, B_1), (A_2, B_2)$ where the A_1, A_2, B_1, B_2 are pairwise disjoint. If $U_i, V_i, i = 1, 2$ are pairwise disjoint neighborhoods of $A_i, B_i, i = 1, 2$ respectively, we can find $g_1, g_2 \in G$ such that $g_i(X \setminus U_i) \subseteq V_i$, $i = 1, 2$. These constitute a ping-pong pair. □

Lemma 3.3. *Let G be a compact topological group and $g \in G$. For every neighborhood U of the identity there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $g^{n_j} \in U$ for all $j \in \mathbb{N}$.*

Proof. Let μ be the normalized Haar measure on G, so that μ is a probability measure of complete support on G and left multiplication by g preserves μ. Let $V \subseteq U$ be an open neighborhood of the identity such that $V \cdot V^{-1} \subseteq U$. Since $\mu(V) > 0$, Poincaré's recurrence theorem implies that there is a sequence $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\mu(V \cap g^{n_j} V) > 0$ for all $j \in \mathbb{N}$. In particular $g^{n_j} \in V \cdot V^{-1} \subseteq U$ for all $j \in \mathbb{N}$. □

The core of the proof of Theorem A is the following statement, which uses ideas from [LBMB22, Proposition 5.2] and implies the extreme contraction properties required by Lemma 3.2. If $g \in \mathrm{AAut}(\mathcal{T})$ we use the same notation as Corollary 2.4, so we write $U_g \subseteq \partial\mathcal{T}$ for the stable set of g and $\mathrm{Per}_{\mathrm{hyp}}(g)$, $\mathrm{Per}_{\mathrm{rep}}(g)$ for the hyperbolic and repelling points of g respectively.

Recall that we denote by $2^{\partial\mathcal{T}}$ the compact metrizable space of all closed subsets of $\partial\mathcal{T}$ equipped with the Vietoris topology.

Proposition 3.4. *Let $H \leq \mathrm{AAut}(\mathcal{T})$ and assume that $\bigcap_{h\in H} U_h$ is empty. Then there exists a finite set $B \subset \partial\mathcal{T}$ such that for every $\varepsilon > 0$ there is a $h \in H$ with $h(\partial\mathcal{T} \setminus B^\varepsilon) \subseteq B^\varepsilon$.*

Proof. By compactness there exist elements $h_1, \ldots, h_k \in H$ such that $\bigcap_{1\leq j\leq k} U_{h_j}$ is empty. By taking powers of the h_j we can assume that every $h_j\big|_{U_j}$ is an isometry. Set $B = \bigcup_{1\leq j\leq k} \mathrm{Per}_{\mathrm{hyp}}(h_j)$ and $C_1 = \partial\mathcal{T} \setminus B^\varepsilon$.

Now h_1 restricted to U_{h_1} is an isometry. Since isometry groups of compact spaces are compact for the compact-open topology, Lemma 3.3 and a diagonal argument shows that there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\sup_{x\in U_{h_1}} d\left(h_1^{n_j}(x), x\right) \leq \frac{1}{j}$$

for all $j \in \mathbb{N}$.

For any closed $D \subseteq \partial\mathcal{T}$ denote by $\overline{\mathrm{Orb}_H(D)}$ the closure of the H-orbit of D in $2^{\partial\mathcal{T}}$. By taking a limit point of $\{h_1^{n_j}(C_1)\}_{j\in\mathbb{N}}$ in $2^{\partial\mathcal{T}}$ we obtain a closed $C_2 \in \overline{\mathrm{Orb}_H(C_1)}$ with

$$C_2 \subseteq (C_1 \cap U_{h_1}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_1).$$

Lemma 3.3 applied to h_2 gives again the existence of a sequence $\{n'_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\sup_{x\in U_{h_2}} d\left(h_2^{n'_j}(x), x\right) \leq \frac{1}{j}$$

for all $j \in \mathbb{N}$. The finite set $\mathrm{Per}_{\mathrm{rep}}(h_2)$ can only intersect C_2 in $\mathrm{Per}_{\mathrm{hyp}}(h_1)$, so by taking again a limit point of $\{h_2^{n'_j}(C_2)\}_{j\in\mathbb{N}}$ we obtain a closed $C_3 \in \overline{\mathrm{Orb}_H(C_2)} \subseteq \overline{\mathrm{Orb}_H(C_1)}$ with

$$C_3 \subseteq (C_2 \cap U_{h_2}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_2) \subseteq (C_1 \cap U_{h_1} \cap U_{h_2}) \sqcup (\mathrm{Per}_{\mathrm{hyp}}(h_1) \cup \mathrm{Per}_{\mathrm{hyp}}(h_2)).$$

Notice that, again, the finite set $\mathrm{Per}_{\mathrm{rep}}(h_3)$ can only intersect C_3 in $\mathrm{Per}_{\mathrm{hyp}}(h_1) \cup \mathrm{Per}_{\mathrm{hyp}}(h_2)$.

By iterating this argument we produce, for every $j = 2, \dots, k+1$, a closed subset $C_j \in \overline{\mathrm{Orb}_H(C_{j-1})} \subseteq \overline{\mathrm{Orb}_H(C_1)}$ such that

$$C_j \subseteq (C_{j-1} \cap U_{h_{j-1}}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_{j-1}) \subseteq (C_1 \cap U_{h_1} \cap \cdots \cap U_{h_{j-1}}) \sqcup \left(\bigcup_{1\le i\le j-1} \mathrm{Per}_{\mathrm{hyp}}(h_i) \right).$$

In particular $C_{k+1} \subseteq \bigcup_{1\le i\le k} \mathrm{Per}_{\mathrm{hyp}}(h_i) = B$, and we are done. □

Proof of Theorem A. Take a subgroup $H \le \mathrm{AAut}(\mathcal{T})$ and suppose that H does not preserve a probability measure on $\partial\mathcal{T}$. If every finitely generated subgroup of H preserves a probability measure on $\partial\mathcal{T}$, the compactness of $\mathrm{Prob}(\partial\mathcal{T})$ equipped with the weak-$*$ topology implies that H preserves a probability measure on $\partial\mathcal{T}$. We may assume then that H is finitely generated.

Suppose that the set $U_H = \bigcap_{h\in H} U_h$ is nonempty. Since every U_h is dynamically defined (see the definition of U_h in Corollary 2.4), we have that U_H is H-invariant. The closed set U_H may be written as $\partial\mathcal{T}'$ for some rooted subtree $\mathcal{T}' \subseteq \mathcal{T}$, and we equip U_H with the restriction of the visual metric of $\partial\mathcal{T}$, which coincides with the visual metric of $\partial\mathcal{T}'$. Every element of H has a proper power that acts as an isometry on $\partial\mathcal{T}'$, so the action of H on $\partial\mathcal{T}'$ is by elliptic almost automorphisms of $\mathcal{T}'$. Since H is finitely generated, Proposition 2.1 shows that the image of H in $\mathrm{AAut}(\mathcal{T}')$ is relatively compact. Thus H preserves a measure on $\partial\mathcal{T}' \subseteq \partial\mathcal{T}$, a contradiction. We conclude that U_H is empty, and in this case Proposition 3.4 and Lemma 3.2 imply that there exists a ping-pong pair for the action of H. □

Remark. The proof of Theorem A shows a slightly stronger statement, namely that for any finitely generated $H \le \mathrm{AAut}(\mathcal{T})$, either H contains a ping-pong pair or there exists a nonempty closed set $C \subseteq \partial\mathcal{T}$ such that the action of H on C is equicontinuous.

Recall that $V_{\mathcal{T}}$ denotes the Higman-Thompson group associated to $\mathcal{T}$.

Proof of Theorem B. Fix a linear order on $\partial\mathcal{T}$ as in the definition of $V_{\mathcal{T}}$. Take $H \le V_{\mathcal{T}}$ a finitely generated subgroup and suppose that H acts without finite orbits on $\partial\mathcal{T}$. If $U_H = \bigcap_{h\in H} U_h$ is nonempty, consider again a rooted subtree $\mathcal{T}' \subseteq \mathcal{T}$ such that $U_H = \partial\mathcal{T}'$, so H acts on U_H by elliptic Higman-Thompson elements of $\mathrm{AAut}(\mathcal{T}')$ for the induced order on $\partial\mathcal{T}'$. Corollary 2.2 shows that the image of H in $\mathrm{AAut}(\mathcal{T}')$ is finite, and thus there exists a finite H-orbit in U_H. This is a contradiction, hence U_H is empty. Again Proposition 3.4 and Lemma 3.2 show that there exists a ping-pong pair for the action of H. □

4. Open questions

We finish with some hopefully tractable open questions.

Question. R. Grigorchuk, V. Nekrashevych and V. Sushchansky introduce in [GNS00] the group $\mathcal{R}$ of homeomorphisms of Cantor space defined by asynchronous transducers, which is known [BB17] to contain the higher-dimensional Brin-Thompson groups from [Bri04]. On the other hand, the automorphism group of a Higman-Thompson group $V_{d,k}$ coincides with the subgroup of bi-synchronizing transducers inside $\mathcal{R}$ [BCM+19]. Do any of these groups satisfy the dynamical Tits alternative?

Question. Let G be a group of homeomorphisms of a compact topological space X such that its groupoid of germs is hyperbolic in the sense of V. Nekrashevych, see [Nek15]. Does the action of G on X satisfy the dynamical Tits alternative? Theorem A shows that this is true for the groups V_G, whose groupoid of germs is hyperbolic whenever G is a contracting self-similar group.

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