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Aspects dynamiques et probabilistes des actions proximales de groupe

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Résumé

Cette thèse est consacrée à l'étude des actions proximales de groupes dénombrables en dimension un et zéro depuis la perspective de la dynamique et des probabilités. Les résultats sont de nature assez variée. Un fil conducteur, parmi d'autres, est la présence des groupes de Thompson F, T et V; presque la totalité des résultats s'appliquent à l'un de ces exemples, et sont parfois motivés par ceux-ci.

Le Chapitre 2 donne une nouvelle preuve d'une version dynamique de l'alternative de Tits pour le groupe de Thompson V, qui se généralise aisément aux groupes de presque automorphismes d'un arbre enraciné localement fini agissant sur son bord.

Le Chapitre 3 montre une version probabiliste de l'alternative de Tits dynamique, due à G. Margulis, pour les groupes agissant minimalement et proximalement par difféomorphismes de S^1, dont la preuve est inspiré par le même phénomène pour marches aléatoires sur une groupe linéaire non virtuellement résoluble. La méthode requiert des hypothèses de moment sur les mesures de probabilité pertinantes, et un énoncé plus faible (et plus simple de prouver) reste vrai sans hypothèses de moment pour des sous-groupes d'homéomorphismes de S^1.

Dans le Chapitre 4 on étudie la question de déterminer sous quelles conditions le bord de Poisson d'un groupe dénombrable agissant sur S^1 proximalement et minimalement, équipé avec une mesure de probabilité, s'identifie avec S^1 muni de sa unique mesure stationnaire. Nous démontrons que c'est jamais le cas si l'action n'est pas topologiquement libre et la mesure de probabilité sur le groupe a entropie de Shannon finie. Ceci s'applique au groupe de Thompson T, pour lequel on peut donner une preuve alternative et plus courte qui se généralise à des groupes de homéomorphismes affines par morceaux de S^1, et répond à une question de B. Deroin et d'Andrés Navas. Ce travail est issu d'une collaboration avec Cosmas Kravaris et Eduardo Silva.

Le Chapitre 5 étudie l'existence de bords stationnaires sur l'espace de sous-groupes moyennables d'un groupe dénombrable, motivé par des liens avec la C*-simplicité de groupes dénombrables. On trouve un critère sur la dynamique du groupe agissant par conjugaison sur son espace de sous-groupes moyennables pour assurer l'existence d'un bord dans tel espace (pour une certaine mesure de probabilité sur le groupe) et on l'applique à des produits en couronne et au groupe de Thompson F. On exhibe une vaste classe de groupes exhibant de la courbure non positive et qui n'admettent pas de tels bords pour n'importe quelle mesure de probabilité. Ce travail est issu d'une collaboration avec Anna Cascioli et Eduardo Silva.

Finalement, le Chapitre 6 étudie la classe de groupes dénombrables tels que la relation de conjugaison entre ses actions minimales par homéomorphismes de la droite admet une transversale Borélienne. On démontre que cette classe est close pour certaines opérations entre groupes, et qu'elle inclut tous les groupes de type fini qui se plongent dans le groupe d'homéomorphismes projectifs par morceaux de la droite. Ce travail est issu d'une collaboration avec Joaquín Brum et Nicolás Matte Bon.

Abstract

This thesis is devoted to the study of proximal actions of countable groups in dimensions one and zero from the perspective of dynamics and probability. The character of the results is fairly diverse. A common theme, among others, is the presence of Thompson's groups F, T and V; almost all the results can be applied to one of these examples, and are sometimes motivated by them.

Chapter 2 gives a new proof of a dynamical version of the Tits alternative for Thompson's group V, which generalizes to groups of almost automorphismes of a rooted locally finite tree acting on its boundary.

Chapter 3 shows a probabilistic version of the dynamical Tits alternative, due to G. Margulis, for groups acting minimally and proximally by diffeomorphisms of S^1, whose proof is inspired by the same phenomenon for random walks on non-virtually solvable linear groups. The method requires moment hypotheses on the corresponding probability measures, and a weaker statement (which is easier to prove) holds true without moment hypotheses on subgroups of homeomorphisms of S^1.

In Chapter 4 we study the question of determining under which conditions the Poisson boundary of a countable group acting minimally and proximally on S^1, endowed with a probability measure, can be identified with S^1 equipped with its unique stationary measure. We show that this is never the case if the action is not topologically free and the probability measure on the group has finite Shannon entropy. The result applies to Thompson's group T, for whom we can give an alternative and shorter proof which generalizes to groups of piecewise affine homeomorphisms of S^1, and answers a question of B. Deroin and of A. Navas. This work is the result of a collaboration with Cosmas Kravaris and Eduardo Silva.

Chapter 5 studies the existence of stationary boundaries with values on the space of amenable subgroups of a countable group, motivated by links with C*-simplicity of countable groups. We find a criterion for the dynamics of the group acting by conjugation on its space of amenable subgroups that ensures the existence of a boundary on this space (for a certain probability measure on the group) and we apply it to wreath products and to Thompson's group F. We exhibit a large class of groups exhibiting non-positive curvature that do not admit such boundaries for any probability measure. This work is the result of a collaboration with Anna Cascioli and Eduardo Silva.

Finally, Chapter 6 studies the class of countable groups such that the conjugacy relation

between their minimal actions by homeomorphisms of the real line admits a Borel transversal. We show that this class is closed under natural group-theoretic operations, and that it contains all finitely generated groups that embed into the group of piecewise projective homeomorphisms of the line. This work is the result of a collaboration with Joaquín Brum and Nicolás Matte Bon.

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Chapter 1

Introduction

This chapter contains an overview of the general context surrounding my mathematical activity, an outline of the objects that appear recurrently in my results along with a brief presentation of some of these, and a list of related open questions. The subsequent chapters are revised versions of several papers or preprints written by myself (Chapters 2 and 3) or in collaboration with others (Cosmas Kravaris and Eduardo Silva for Chapter 4, Anna Cascioli and Eduardo Silva for Chapter 5 and Joaquín Brum and Nicolás Matte Bon for Chapter 6), and are self-contained. Generative AI tools have not played a relevant role in the work presented in this thesis.

1.1 Context and results

Geometric group theory is the study of finitely generated groups G as coarse geometric objects through the collection of word distances d_S coming from all finite generating sets $S \subseteq G$. Though originally drawing from combinatorial group theory, it was born as a subject following work of Gromov, Milnor, Mostow, Stallings and many others in the second half of the 20th century. A distinguished line of inquiry in this theory is the study of groups exhibiting some degree of non-positive curvature or admitting a rich enough action on some non-positively curved space, motivated by the families of fundamental groups of hyperbolic surfaces or 3-manifolds and other groups arising in geometric topology or differential geometry (e.g. mapping class groups of such surfaces or lattices in semisimple Lie groups).

The main object of study of this thesis is the class of countable groups acting proximally on X by homeomorphisms or diffeomorphisms, where $X \in \{\mathbb{R}, S^1, \{0,1\}^{\mathbb{N}}\}$. Here, a group action $G \curvearrowright X$ is said to be *proximal* if for every $x, y \in X$ there exists $z \in X$ and $(g_n)_{n\geq 0} \subseteq G$ such that $(g_n.x)_{n\geq 0}$ and $(g_n.y)_{n\geq 0}$ converge to z. Although many groups of geometric origin fall into this family (for instance, $\mathrm{PSL}_2(\mathbb{R})$ and its discrete subgroups act proximally on the projective space of lines of $\mathbb{R}^2$, which is S^1), the class we consider does not sit *a priori* inside this framework. The comparative simplicity of the spaces on which these groups act, as opposed to higher-dimensional manifolds, allows for a rather large supply of examples and non-examples: as an extreme instance of the former, any countable group acts faithfully on the Cantor space by considering the full-shift on itself, while restrictions related to the left-orderability or circular-orderability of groups acting faithfully on 1-dimensional manifolds illustrate the latter.

As a consequence the richness of this class of groups comes from variety of behaviours that they can accomodate: for instance, a recurrent feature is the possibility of having a proximal and micro-supported action on the space X. Here, a group action $G \curvearrowright X$ by homeomorphisms is said to be *micro-supported* if for every non-empty open set $U \subseteq X$ the rigid stabilizer $G_U = \{g \in G : g\big|_{X\setminus U} = \mathrm{id}_{X\setminus U}\}$ is non-trivial. The contrast with the aforementioned groups of geometric origin lies in the fact that while many of these do admit proximal actions on compact spaces (a hyperbolic group acting on its Gromov boundary or a non-virtually solvable linear group acting irreducibly on projective space are the prime examples), these actions tend to be *topologically free*, that is, such that the set of fixed points of any non-trivial element has empty interior. Another way to make this contrast apparent is the following situation: suppose that $G \curvearrowright X$ is micro-supported, and that there is an non-empty open $U \subseteq X$ and $g \in G$ such that the $g^n(U), n \in \mathbb{Z}$ are pairwise disjoint. Then $\langle g, G_U\rangle$ is the wreath product $G_U \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} G_U \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on $\bigoplus_{\mathbb{Z}} G_U$ by shifting coordinates. Even though this subgroup has no reason to be well embedded in G, many results of this thesis (notably from Chapter 4 onwards) exploit phenomena occuring in lamplighters.

In what follows, when X is a 1-manifold we denote by $\mathrm{Homeo}_0(X)$ the group of orientation-preserving homeomorphisms of X, and similarly with the orientation-preserving diffeomorphisms $\mathrm{Diff}_0(X)$ and piecewise affine homeomorphisms with finitely many breakpoints $\mathrm{PAff}_0(X)$ (for some fixed, unmentioned affine structure on X).

Thompson groups

The Thompson groups F, T and V are finitely presented groups introduced by R. Thompson in 1965 [Tho65]. We refer to [CFP96] for a more complete introduction to these groups.

To define them, consider two partitions $\mathcal{P}_1, \mathcal{P}_2$ of $\{0,1\}^{\mathbb{N}}$ into an equal number of *cylinders*, that is, clopen sets of the form

$$[w] = \{z \in \{0,1\}^{\mathbb{N}} : z_{[0,|w|-1]} = w\}$$

where w is a finite word on the alphabet $\{0,1\}$, of length $|w|$. We identify the cylinders in the $\mathcal{P}_i$ with the finite words they are specified by. Hence for every $z \in \{0,1\}^{\mathbb{N}}$ we may write the infinite word z as the concatenation $w(z)z_+$ where $w(z)$ is the unique finite word in $\mathcal{P}_1$ such that $z \in [w(z)]$. If $\sigma \colon \mathcal{P}_1 \to \mathcal{P}_2$ is a bijection we obtain a homeomorphism of $\{0,1\}^N$ by setting $\widetilde{\sigma}(z) = \sigma(w(z))z_+$ for every $z \in \{0,1\}^{\mathbb{N}}$.

The family of all homeomorphisms of $\{0,1\}^{\mathbb{N}}$ obtained in this way is Thompson's group V, which locally respects the lexicographic linear order $<$ on $\{0,1\}^{\mathbb{N}}$ coming from the obvious orders on every copy of $\{0,1\}$. The requirement that the homeomorphisms respect the order $<$ (resp. the circular order on $\{0,1\}^{\mathbb{N}}$ induced by $<$) defines Thompson's group F (resp. Thompson's group T).

The groups T and V are simple and contain non-abelian free groups, while F has simple commutator $[F, F]$ with quotient $F/[F, F] \cong \mathbb{Z}^2$ and, even though it contains no non-abelian free groups [BS85], its amenability is an outstanding open problem. A positive answer would show that F is a natural example of an amenable group that is not elementarily amenable (that is, that cannot be constructed from finite or abelian groups through group extensions and direct unions), and a negative answer would show that F is a natural example of a non-amenable group without non-abelian free groups. At the time of its definition, T was the first example of an infinite, simple and finitely presented group.

When dealing with F and T we will use their realizations as groups of piecewise dyadically affine homeomorphisms of the interval and the circle, respectively. That is, an element $f \in \mathrm{Homeo}_0([0, 1])$ belongs to F if and only if there is a finite subset $\mathcal{B} \subseteq [0, 1]$ such that on every connected component of $[0, 1] \setminus \mathcal{B}$, f is of the form $x \mapsto 2^k + q$ for some $k \in \mathbb{Z}$, $q \in \mathbb{Z}[1/2]$. The realization for T is the same replacing $[0, 1]$ by $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}[1/2]$ by $\mathbb{Z}[1/2]/\mathbb{Z}$, and both follow from the fact that partitions of $\{0, 1\}^{\mathbb{N}}$ into clopen sets correspond naturally to partitions of $[0, 1]$ and of $\mathbb{R}/\mathbb{Z}$ into dyadic intervals. The commutator $[F, F]$ corresponds in this realization to the elements of F that have derivative equal to 1 near the endpoints of $[0, 1]$. By conjugating this representation of F by a homeomorphism $h\colon (0, 1) \to \mathbb{R}$ sending $\mathbb{Z}[1/2] \cap (0, 1)$ to $\mathbb{Z}[1/2]$, we see that F is the group of piecewise dyadically affine homeomorphisms of $\mathbb{R}$ which are translations on neighborhoods of $\pm\infty$.

There is a vast literature on combinatorial, homological and algebraic approaches to the study of the Thompson groups and their many variations, which yield results on their finiteness properties, their metric properties, their automorphism groups or their subgroups (see the references in [FH24, BCM+24, GP25], for instance). Our methods rely mostly on the dynamics of their defining actions instead. An abstract reason why this might be possible is a result of M. Rubin [Rub89], asserting that a group may admit at most one sufficiently rich micro-supported action on a locally compact space. Concretely, a group action $G \curvearrowright X$ on a Hausdorff, locally compact and perfect space X is said to be *Rubin* if for every $x \in X$ and open neighborhood $U \subseteq X$ of x the closure of the orbit $G_U.x$ contains a neighborhood of x. Then Rubin's theorem says that if $G \curvearrowright X$, $G \curvearrowright Y$ are Rubin actions, then X and Y are G-equivariantly homeomorphic.

Dynamical Tits alternatives

Several groups of geometric origin satisfy the Tits alternative (see the references in [dlH00, Section II B, Complement 42]), which states that their subgroups are either virtually solvable or contain non-abelian free groups. While many groups of homeomorphisms such as Thompson's group F or the first Grigorchuk group do not satisfy this alternative, a fruitful replacement is the following: a group G acting on a compact space X is said to verify the *dynamical Tits alternative* if for every subgroup $H \subseteq G$, either the action of H on X preserves a probability measure or H contains a *ping-pong pair*, that is, two elements $g, h \in H$ such that there are disjoint open sets $U_1, U_2, V_1, V_2 \subseteq X$ with $g(X \setminus U_1) \subseteq V_1$ and $h(X \setminus U_2) \subseteq V_2$. In the latter case, the group generated by g, h is free by the Klein ping-pong lemma.

Examples of groups satisfying this dichotomy are Thompson's group V [HM19] and the homeomorphism group of the circle [Mar00]. Our first result gives a new proof of the statement for V that generalizes to the group of almost-automorphisms of a locally finite rooted tree $\mathcal{T}$, which relies on the dynamics of individual almost automorphisms following [GL21, SD10]. At least in the case when $\mathcal{T}$ is the infinite binary rooted tree, the group of almost-automorphisms $\mathrm{AAut}(\mathcal{T})$ is known as the Neretin group [Ner92] and is a locally compact simple subgroup of $\mathrm{Homeo}(\partial\mathcal{T})$ generated by V and the rooted tree automorphisms of $\mathcal{T}$. See Chapter 2 below and [GL18] for more background on these groups.

Theorem (Chapter 2)**.** *Let $\mathcal{T}$ be a locally finite rooted tree. Then $\mathrm{AAut}(\mathcal{T})$ satisfies the dynamical Tits alternative.*

Moreover, if H is a finitely generated subgroup of $\mathrm{AAut}(\mathcal{T})$ that locally preserves a linear order on $\partial\mathcal{T}$, then either the action of H on $\partial\mathcal{T}$ has a finite orbit or H admits a ping-pong pair.

There are several strengthenings of the Tits alternative of a qualitative or quantitative nature. An elementary one states that when G is a connected non-solvable Lie group, the set of pairs in $G \times G$ that generate a non-abelian free group has full Haar measure [Eps71]. An analoguous statement for homeomorphisms of the circle is the fact that the set of pairs in $\mathrm{Homeo}(S^1) \times \mathrm{Homeo}(S^1)$ that generate a non-abelian free group is Baire-generic [Ghy01, Proposition 4.5].

We will be interested in probabilistic strengthenings of the Tits alternative, for which we need some notation. A probability measure μ on a group G is said to be *non-degenerate* if the semigroup generated by the support of μ is all of G. In this case, we denote by $(f_\omega^n)_{n\geq 0} \in G^{\mathbb{N}}$ the *left random walk driven by* μ, where $f_\omega^N = f_{\omega_N} \circ f_{\omega_{N-1}} \circ \cdots \circ f_{\omega_0}$ for every $N \in \mathbb{N}$ and $(f_{\omega_n})_{n\geq 0}$ is a random variable on $G^{\mathbb{N}}$ with distribution $\mu^{\otimes\mathbb{N}}$. A result of R. Aoun [Aou11, Aou13] asserts, in one of its forms, that if G is a non-virtually solvable linear group over $\mathbb{R}$ and μ_1, μ_2 are non-degenerate measures on G such that each $\int_{G_i} \|g\|^\delta \,\mathrm{d}\mu_i(g)$ is finite for some $\delta > 0$ (where $\|\cdot\|$ is any matrix norm), then $\mu_1^{\otimes\mathbb{N}} \otimes \mu_2^{\otimes\mathbb{N}}$-almost surely there is an $N \in \mathbb{N}$ such that for all $n \geq N$, the left random walk f_ω^n driven by μ_1 and the left random walk $f_{\omega'}^n$ driven by μ_2 generate a non-abelian free group. The main result of Chapter 3 is a version for circle diffeomorphisms that follows a similar strategy as Aoun's results, and applies in particular to Thompson's group T by [GS87].

Theorem (Chapter 3)**.** *Let $G_1, G_2 \subseteq \mathrm{Diff}_0^1(S^1)$ be a countable groups acting proximally on S^1 and μ_1, μ_2 be non-degenerate measures on G_1, G_2 respectively such that the integrals*

$$\int_{G_i} \max\left\{\|g'\|_\infty, \left\|(g^{-1})'\right\|_\infty\right\}^\delta \mathrm{d}\mu_i(g)$$

are finite for some $\delta > 0$. Then $\mu_1^{\otimes\mathbb{N}} \otimes \mu_2^{\otimes\mathbb{N}}$-almost surely there is an $N \in \mathbb{N}$ such that for all $n \geq N$, the left random walk f_ω^n driven by μ_1 and the left random walk $f_{\omega'}^n$ driven by μ_2 generate a non-abelian free group.

After the appearance of the preprint [GV25], I. Choi in [Cho25] generalized the previous theorem to measures μ_i supported on $\mathrm{Homeo}_0(S^1)$ with no moment conditions by adapting S. Gouëzel's pivoting technique [Gou22] to this setting.

Poisson boundaries

Given a probability measure μ on a group G, a measurable action of G on a standard probability space (X,ν) is said to be *μ-stationary* if $\nu = \sum_{g\in G}\mu(g)g_*\nu$. The Poisson boundary of a locally compact group G equipped with a probability measure μ is a μ-stationary system $(\partial_\mu G, \nu_\mu)$ introduced by H. Furstenberg [Fur63] that parametrizes all μ-harmonic measures on G in a fashion that generalizes the Poisson integral formula for harmonic functions on the disk. Other characterizations, in terms of the stationary dynamics of (G,μ) and in terms of the asymptotic events of the right random walk on G driven by μ, explain the central role of this object in rigidity problems [Fur71, Mar91, Zim84] and in the study of growth of groups [EZ20] (see [BF14] and [Ers10] respectively, and the references therein). These developments have followed from contributions by Furstenberg, Vershik, Kaimanovich, Derrienic, Margulis, Zimmer and many more.

A common theme in this theory is, given a countable group G, to identify $(\partial_\mu G, \nu_\mu)$ with natural μ-stationary G-systems constructed from combinatorial, geometric or algebraic features of G. The usual tool to do so is Kaimanovich's conditional entropy criterion [Kai85, Kai00], a numerical condition which requires the measure μ to have finite Shannon entropy $H(\mu) = -\sum_{g\in G}\mu(g)\log\mu(g)$ (the necessity of this condition was recently established in [CF25]). To mention two exemplary instances of this kind of result, denote by $\mathbf{w} = (w_n)_{n\geq 0}$ the *right random walk driven by* μ, where $w_N = g_0 g_1 \cdots g_N$ for every $N \in \mathbb{N}$ and $(g_n)_{n\geq 0} \in G^{\mathbb{N}}$ is a random variable with distribution $\mu^{\otimes\mathbb{N}}$.[1]

- Let G be a Gromov-hyperbolic group and μ a non-degenerate measure on G. A qualitative argument originally due to H. Furstenberg in the linear setting [Fur63] shows that almost surely the right random walk $\mathbf{w} = (w_n)_{n\geq 0}$ converges to a random point $\xi(\mathbf{w})$ in the Gromov boundary $\partial_{\mathrm{Gr}}G$. The correspondence $\mathbf{w} \mapsto \xi(\mathbf{w})$ defines a μ-stationary measure on $\partial_{\mathrm{Gr}}G$, which is known to be a model for the Poisson boundary of (G,μ) when different finiteness conditions are imposed on μ [Anc87, Kai00, CFFT25].

- Let $G = A \wr B$ be a wreath product of countable groups A, B where A is non-trivial and B is infinite. Let μ be a non-degenerate finitely supported measure such that the pushforward of μ to B induces a transient random walk. If we write $w_n = (\varphi_n, b_n) \in \bigoplus_B A \rtimes B$ for every $n \in \mathbb{N}$, then [KV83] shows that the configurations $(\varphi_n)_{n\geq 0}$ stabilize to an random function $\varphi(\mathbf{w}) \in A^B$. Again, this correspondence defines a μ-stationary measure on A^B, which is known to coincide with the Poisson boundary of (G,μ) by imposing finiteness conditions on μ or geometric conditions on B [Ers11, LP21, FS23].

If $G \subseteq \mathrm{Homeo}_0(S^1)$ is a countable group acting proximally on S^1 with no finite orbits, then [DKN07] shows that for any $\mu \in \mathrm{Prob}(G)$ there is a unique μ-stationary probability measure ν on S^1. Since (S^1,ν) is known to coincide with the Poisson boundary of (G,μ) when μ is a finite entropy measure on a lattice G of $\mathrm{PSL}_2(\mathbb{R})$, it is natural to ask whether this holds for any

[1] The difference in notation between left and right random walks owes to differing conventions in the communities studying random dynamical systems and random walks on groups rather than to a conscious decision.

countable group G acting proximally on S^1 with no finite orbits. The proof of the dynamical Tits alternative in [Mar00] shows that one may assume that the action of G on S^1 is minimal, and we give a general obstruction for (S^1, ν) to be the Poisson boundary of (G, μ) in this case.

Theorem (Chapter 4)**.** *Let $G \subseteq \mathrm{Homeo}_0(S^1)$ be a countable group acting minimally, proximally and non-topologically freely on S^1, and let $\mu \in \mathrm{Prob}(G)$ be a non-degenerate measure with finite entropy. Then (S^1, ν) is not the Poisson boundary of (G, μ).*

The previous theorem applies in particular to Thompson's group T, answering questions posed by B. Deroin [Der13b] and A. Navas [Nav18]. It also contrasts with the positive results in this direction proven in [Der13b]. The proof adapts a method of A. Erschler [Ers04] (originally employed to show that a large class of random walks on wreath products have non-trivial Poisson boundary) to a conditional entropy setting. An alternative, slightly shorter proof is given in Section 4.8 below for groups of piecewise affine homeomorphisms of $\mathbb{R}/\mathbb{Z}$, and uses a specific quotient of the Poisson boundary first constructed by V. Kaimanovich for Thompson's group F [Kai17]. Both rely on analogies with wreath products.

Quotients of the Poisson boundary $(\partial_\mu G, \nu_\mu)$ are called *μ-boundaries*, and are a special kind of ergodic μ-stationary systems with opposite behaviour than measure-preserving systems. Our next results study the existence of μ-boundaries in the *Chabauty space* $\mathrm{Sub}(G)$ of subgroups of G, which is a compact metrizable space when equipped with the topology generated by the sets

$$\{H \in \mathrm{Sub}(G) : Q \cap H = Q \cap K\},\ K \in \mathrm{Sub}(G),\ Q \subseteq G \text{ finite}$$

on which G acts by homeomorphisms through conjugation.

We show that Thompson's group F admits a μ-boundary on its space of amenable subgroups $\mathrm{Sub}_{\mathrm{am}}(F)$, which is a closed subspace of $\mathrm{Sub}(F)$. This state of affairs contrasts with the fact that all ergodic invariant measures and minimal closed subsystems of $\mathrm{Sub}(F)$ come from normal subgroups [DM14, LBMB18].

Theorem (Chapter 5)**.** *There is a symmetric and fully-supported measure $\mu \in \mathrm{Prob}(F)$ with finite entropy such that $\mathrm{Sub}_{\mathrm{am}}(F)$ admits a non-atomic μ-boundary.*

The proof uses the notion of record times introduced in [FHTVF19] and systematized in [EK23], along with a dynamical criterion inspired from the case of wreath products (that admit many measures with the properties mentioned in the theorem statement, see the introduction of Chapter 5).

One motivation to consider dynamics on the Chabauty space, and particularly on its closed space of amenable subgroups, is a theorem of M. Kennedy [Ken20] (drawing from his work with Breuillard-Kalantar-Ozawa [KK17, BKKO17]) stating that a countable group G is *C*-simple*, a strong form of non-amenability that corresponds to the absence of closed non-trivial ideals in the reduced C*-algebra $C^*_{\mathrm{red}}(G)$, if and only if there are no closed minimal subsystems of $\mathrm{Sub}_{\mathrm{am}}(G)$. A formulation in terms of stationary dynamics is given by Hartman-Kalantar in [HK23], and proves that this condition is equivalent to the existence of a non-degenerate $\mu \in \mathrm{Prob}(G)$ such that the action $G \curvearrowright \mathrm{Sub}_{\mathrm{am}}(G)$ admits a unique μ-stationary measure, namely the Dirac mass on the trivial subgroup.

For some classes of C*-simple groups exhibiting non-positive curvature, [HK23] shows that any non-degenerate $\mu \in \mathrm{Prob}(G)$ verifies this property. The previous theorem says that F cannot verify this extreme form of C*-simplicity (which is relevant since F is non-amenable if and only if it is C*-simple [LBMB18]). In contrast, Theorem J below shows that a large class of groups G exhibiting some degree of non-positive curvature do not admit non-degenerate $\mu \in \mathrm{Prob}(G)$ such that $\mathrm{Sub}_{\mathrm{am}}(G)$ supports a non-atomic μ-boundary.

Theorem (Chapter 5)**.** *Let G be a countable group, $\mu \in \mathrm{Prob}(G)$ be non-degenerate and let η be a μ-boundary on $\mathrm{Sub}(G)$ that is not a Dirac mass on a finite normal subgroup. Then η gives full measure to normalish subgroups of G.*

Here, we say a subgroup H of G is *normalish* if $\bigcap_{z \in Z} zHz^{-1}$ is infinite for every finite subset $Z \subseteq G$. This notion was introduced in [BKKO17] to give new criteria of C*-simplicity of groups. Several classes of groups are known to not admit amenable normalish subgroups, such as groups with some positive ℓ^2-Betti number [BFS14], non-trivial bounded cohomology or linear groups with trivial amenable radical [BKKO17].

Structure theory for proximal actions on $\mathbb{R}$

While initially motivated by the theory of circle diffeomorphisms and of codimension-one real foliations, the study of countable groups acting on 1-dimensional manifolds by homeomorphisms has developed into an independent subject in the last 30 years following work of Ghys, Matsumoto, Witte-Morris, Navas, Calegari and many others. It now plays a role in questions related to left-orderability [Nav10] and circular-orderability [Cal04] of countable groups, hyperbolic structures on compact surfaces [Mat87, MW24] and 3-manifold topology [BGW13], among other topics (see [DNR16], [Man23] and [CR16, Cal07] respectively, and the references therein). We refer to [Ghy01, Nav11, KK21] for more complete treatments of the subject, and [Nav18] for a survey of problems.

Recall that a group action on a topological space X is said to be *minimal* if all orbits of points of X are dense in X. When G is a countable group, a theorem of É. Ghys [Ghy87] gives a complete invariant (in terms of the bounded cohomology of G) of minimal actions of G by homeomorphisms of S^1 up to conjugacy. The existence of such an explicit invariant for actions up to conjugacy is a result by itself, and can be framed in the study of Borel equivalence relations. S. Matsumoto [Mat86] reformulated Ghys' theorem in more concrete terms, from which one shows that this invariant is *Borel*: that is, there is a Borel map from the space of minimal actions $\mathrm{Rep}_{\min}(G, S^1)$ of G on S^1 to a standard Borel space whose fibers are exactly the conjugacy classes.

When considering minimal actions of G on $\mathbb{R}$ by orientation-preserving homeomorphisms, the existence of a complete invariant depends on G: for instance, the answer is positive for abelian groups [Höl01], more generally for groups with no non-abelian free subsemigroups [Pla75], and for solvable Baumslag-Solitar groups [Riv10]. The answer is negative for the non-abelian free group F_2 (see [BMBRT24, Section 14.4] or Example 6.2.3 below), and we are led to study the following class of groups.

Definition 1.1.1 (Chapter 6)**.** *Define $\mathcal{C}$ as the class of countable groups G such that there is a complete Borel invariant of minimal actions of G on $\mathbb{R}$.*

Given the preceding examples, a natural question is if all amenable (or elementary amenable) groups are in $\mathcal{C}$. We do not know do answer, but our theorems show that the class $\mathcal{C}$ is large.

Theorem (Chapter 6)**.** *Let G be a finitely generated group.*

- *If there is a normal subgroup $H \subseteq G$ such that G/H and H belong to $\mathcal{C}$, then $G \in \mathcal{C}$.*
- *If $(H_n)_{n\geq 0}$ are finitely generated groups in $\mathcal{C}$, then $\bigoplus_{n\geq 0} H_n \in \mathcal{C}$.*
- *If $\Phi\colon H \to K$ is an isomorphism between finite-index subgroups H, K of G which belong to $\mathcal{C}$, then the HNN-extension G^Φ is in $\mathcal{C}$.*

In particular, the class $\mathcal{C}$ contains all finitely generated solvable groups, all Baumslag-Solitar groups $\mathrm{BS}(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ for $m, n \in \mathbb{Z} \setminus \{0\}$ and is closed under wreath products of finitely generated groups.

Theorem (Chapter 6)**.** *Let G be a finitely generated group of piecewise affine homeomorphisms of $\mathbb{R}$. Then $G \in \mathcal{C}$.*

While the previous results go in the direction of the complexity of Borel equivalence relations (as expounded in [BK96, Hjo00]) what may be more relevant for the general context of this thesis is the method of proof. The statement $G \in \mathcal{C}$ is equivalent to a purely dynamical condition on proximal minimal actions $G \curvearrowright^{\varphi} \mathbb{R}$, essentially saying that any sequence of homeomorphisms of $\mathbb{R}$ asymptotically centralizing φ eventually converges to the identity. The verification of this condition in the different cases at hand uses extensively a sort of structure theory for subgroups of $\mathrm{Homeo}_0(\mathbb{R})$.

The proof of the dynamical Tits alternative for $\mathrm{Homeo}_0(S^1)$ from [Mar00] provides more detailed dynamical information for an action of a countable group G on S^1. Indeed, if there is no invariant probability measure for $G \curvearrowright^{\psi} S^1$, up to collapsing wandering intervals and passing to a quotient of the circle by a finite-order homeomorphism centralizing G we obtain an action $G \curvearrowright^{\widetilde{\psi}} S^1$ which is *extremely proximal*, that is, such that for every pair of non-empty open intervals $I, J \subsetneq S^1$ there is $g \in G$ such that $\widetilde{\psi}(g).I \subseteq J$. A similar statement is true for actions on $\mathbb{R}$, but adds the caveat that when G is not finitely generated, it may happen that an action on $\mathbb{R}$ does not admit a closed invariant set where every orbit is dense (i.e. a *minimal set*). This situation appears even when considering finitely generated groups, by restricting actions to infinitely generated subgroups.

One of the contributions of [BMBRT24] is a way to obtain more structure in this context: if $G \curvearrowright^{\varphi} \mathbb{R}$ is a minimal action such that $N \subseteq G$ is a normal subgroup that admits no minimal set on $\mathbb{R}$, then φ is *laminar*, that is, preserves a closed, unbounded and non-crossing set of intervals in $\{(x, y) \in \mathbb{R}^2 : x < y\}$ constructed from $N \curvearrowright^{\varphi} \mathbb{R}$. The normality condition can be considerably weakened so as to be applied to many suitable conjugation-invariant families of subgroups, such as rigid stabilizers of a given action of G on $\mathbb{R}$. Following [BMBRT24, Chapter 9], this gives enough structure on the class of all actions of a micro-supported group to prove the last statement of the previous theorem. We remark that Thompson's F was shown to belong to $\mathcal{C}$ in [BMBRT24, Chapter 16] through structural results on the class of its minimal actions on $\mathbb{R}$ which are unavailable for a general subgroup of $\mathrm{PAff}_0(\mathbb{R})$.

1.2 Open questions

The following are some hopefully tractable open questions that I find stimulating. Some of these are repeated in later chapters.

Concerning Chapter 2:

- The whole homeomorphism group of the Cantor space (or even the group of biLipschitz homeomorphisms of Cantor space) does not verify the alternative.[2] Nevertheless, does the dynamical Tits alternative hold for the higher dimensional Brin-Thompson groups defined in [Bri04] acting on Cantor space? Does it hold for the group of homeomorphisms of the Cantor space defined by asynchronous transducers as in [GNS00] or by bi-synchronizing transducers as in [BCM+24]? If $\mathcal{G}$ is a groupoid of germs of a compact metric space X which is hyperbolic in the sense of V. Nekrashevych [Nek15], does the action of the full group $F(\mathcal{G})$ on X satisfy the alternative?

- Let X be a compact metric space and $\mathbb{R} \curvearrowright^{\Psi} X$ a free, continuous flow. Let $\mathcal{K}$ be the group of homeomorphisms of X that preserve the orbits of Ψ. Does $\mathcal{K} \curvearrowright X$ satisfy the alternative? Through the existence of the *harmonic space* $\mathrm{Harm}(G)$ of a countable group G (see the introduction of Chapter 6), a positive answer would imply a conjecture of P. Linnell asserting that a left-orderable group with no non-abelian free subgroups is locally indicable. The general case seems out of hand at the moment, but the two following variations may be more amenable to analysis. Both were pointed out to me by Nicolás Matte Bon.

  Let h be a self-homeomorphism of the Cantor space and let X be the suspension of h. A basis of the topology of X is given by *flow boxes* $U \subseteq X$ homeomorphic to $K \times I$ where K is the Cantor space and $I \subseteq \mathbb{R}$ is an open interval. Following [MBT20], call a homeomorphism g of X *locally transversally constant* if for sufficiently small flow boxes g is of the form $g\colon (k, x) \mapsto (k, g_U(x))$ where g_U is some homeomorphism between real intervals. Denote by $\mathcal{H}_1$ the group of locally transversally constant homeomorphisms of X. Does $\mathcal{H}_1 \curvearrowright X$ satisfy the dynamical Tits alternative?

  Let $\mathcal{H}_2$ be the group of smooth diffeomorphisms of the torus $S^1 \times S^1$ preserving a (given) irrational linear flow on $S^1 \times S^1$. Does $\mathcal{H}_2 \curvearrowright S^1 \times S^1$ satisfy the dynamical Tits alternative?

[2] Take any non-amenable countable group G without non-abelian free subgroups. Since G is non-amenable, there is a subshift $X \subseteq \{0, 1\}^G$ with no invariant probability measures [FKSV25, Theorem 5.13]. X cannot be countable, so it contains a G-invariant Cantor subset Y. The action $G \curvearrowright Y$ does not satisfy the alternative.

Concerning Chapter 3:

- As a general program, to what extent can methods used to study hyperbolic (resp. non-virtually solvable linear groups) acting on their Gromov boundary (resp. projective space) be adapted to the setting of a group acting proximally on S^1? For hyperbolic groups one can use [Cho25] as a starting dictionary between both worlds. As a test example, does a version of the Abels-Margulis-Soifer lemma [AMS95] (see [KP25] for a version for groups acting on Gromov-hyperbolic spaces by isometries) apply to proximal representations of countable groups on $\mathrm{Homeo}(S^1)$?

  For linear groups, [BM24] employs spectral methods used initially to study random walks on semisimple real Lie groups (in the line of [BQ16]) to deduce limit laws for random walks on subgroups of $\mathrm{Diff}_0^{1+\alpha}(S^1)$ preserving no probability measure on S^1. A fundamental input seems to be the inequality stated in Theorem 3.4.2 below (which is certainly not the first instance of this inequality in the literature). This inequality allows one to prove the following: fix $f_1, \ldots, f_d \in \mathrm{Diff}_0^{1+\alpha}(S^1)$ acting proximally with no global fixed point on S^1, and let ρ be the function assigning to each $\mathbf{p} \in [0,1]^d$ with $\sum_i p_i = 1$ the Lyapunov exponent of the measure $\sum_i p_i \delta_{f_i}$. Then ρ is analytic (with the same proof as in the linear case [Per91]).

Concerning Chapter 4:

- Let μ be a non-degenerate finitely supported measure on Thompson's group T. Does the Poisson boundary of (T, μ) coincide with the breakpoint boundary defined in Section 4.8?[3] Does it coincide with the stationary joining of the breakpoint boundary with (S^1, ν) where ν is the unique μ-stationary measure on S^1? As a strictly weaker version of this question, is (S^1, ν) a quotient of the breakpoint boundary?

- If μ is *any* non-degenerate measure on Thompson's group T, is it still true that (S^1, ν) is not the Poisson boundary of (T, μ)? If true, the proof would have to bypass techniques based on Avez entropy entirely, as in [CF25].

Concerning Chapter 5:

- Theorem J and [BFS14] show that if G admits a μ-boundary on $\mathrm{Sub}(G)$ whose support consists of subgroups with vanishing ℓ^2-Betti numbers, then the ℓ^2-Betti numbers of G vanish. This is well known when the μ-boundary is a Dirac mass on a normal subgroup. Is this true for invariant random subgroups or, more generally, for μ-stationary random subgroups? The answer is positive for an invariant random subgroup supported on amenable subgroups of G by [BDL16], who prove that in this case the invariant random subgroup is supported on subgroups of the amenable radical of G.

[3]The same question for Thompson's group F is also open. In both cases a positive answer seems to encounter the same problems as when extending the results of [FS23] (which make explicit the Poisson boundary of wreath products of groups for a vast class of measures) to permutational wreath products.

- The extension of Corollary I (the existence of non-degenerate $\mu \in \mathrm{Prob}(F)$ admitting non-atomic μ-SRSs on abelian subgroups of F) to Thompson's T and V should be true but appears to require new ideas. There is a canonical procedure to produce IRSs of an overgroup G given IRSs of a group H [KQ19]. A version for SRSs is plausible but faces the problem of choosing a non-degenerate measure on G given one on H.

- As a general program, to what extent can the properties of μ-stationary measures (or μ-boundaries) on the space of subgroups of a countable group be made to reflect known results for uniformly recurrent subgroups? Results of A. Alpeev [Alp25] indicate that the analogy should hold between URSs and μ-stationary measures (or μ-boundaries) for a Baire-generic $\mu \in \mathrm{Prob}(G)$. For instance, Corollary I shows that an exact analogue of the absence of non-trivial URSs of Thompson's F from [LBMB18] fails for certain non-degenerate measures.

- What are the generalized Baumslag-Solitar groups with only countably many amenable subgroups? There are two possible applications of an answer: first, [dlHP11, Proposition 5] states that a Baumslag-Solitar group is C*-simple if and only if its amenable radical is trivial (this is true for GBS groups [MV25, Proposition 9.1]). By [BKKO17, Theorem 6.11], a GBS group G with countably many amenable subgroups satisfies a strengthening of this property, since a (possibly non-finitely generated) subgroup H of G is C*-simple if and only if its amenable radical is trivial. Second, a group with countably many amenable subgroups cannot contain groups with uncountably many amenable subgroups. In both cases, we do not know if this gives new non-finitely generated C*-simple groups acting on trees or new obstructions to embeddings between GBS groups, see [MWY25] and [Lev15] respectively.

Concerning Chapter 6:

- Is the conjugacy relation between minimal actions of a non-finitely generated countable group G by orientation-preserving homeomorphisms of $\mathbb{R}$ essentially hyperfinite? Are all the orbit equivalence relations induced by actions of $\mathrm{Homeo}_0(\mathbb{R})$ essentially countable? Equivalently, does $\mathrm{Homeo}_0(\mathbb{R})$ admit a turbulent action in the sense of G. Hjorth (see [Hjo00, Chapter 3])?

- Are all amenable or elementary amenable groups contained in $\mathcal{C}$? Is there a hyperbolic or acylindrically hyperbolic group in $\mathcal{C}$ that admits a faithful minimal action on $\mathbb{R}$ by homeomorphisms? If $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ is a group in $\mathcal{C}$, is the group of piecewise-G homeomorphisms of $\mathbb{R}$ in $\mathcal{C}$? Can the conditions of finite generation required in Theorem L on closure properties of $\mathcal{C}$ under group-theoretic operations be dispensed with?

- M. Brin observed in [Bri96] that the fact that the standard action of F on $\mathbb{R}$ is highly transitive in an adequate sense implies, by Rubin's theorem [Rub89] (actually a stronger variant by McCleary-Rubin [MR05]), that the automorphism group of F is canonically identified with a subgroup of the normalizer of F in $\mathrm{Homeo}(\mathbb{R})$. The results of [Bri96] describe $\mathrm{Aut}^+(F) = \mathrm{Aut}(F) \cap \mathrm{Homeo}_0(\mathbb{R})$ as a short exact sequence

$$1 \longrightarrow F \longrightarrow \mathrm{Aut}^+(F) \longrightarrow T \times T \longrightarrow 1$$

by showing that $\mathrm{Aut}^+(F)$ is the group of piecewise dyadically affine homeomorphisms of $\mathbb{R}$ that eventually commute with the translation $x \mapsto x + 1$ and have a discrete set of breakpoints. Since both F and $T \times T$ are in $\mathcal{C}$ and $\mathrm{Aut}^+(F)$ is finitely generated (see [BC13] for a finite presentation) we conclude that $\mathrm{Aut}^+(F)$ is in $\mathcal{C}$. To which degree does this hold true for a the automorphism group of finitely generated piecewise affine group G acting in a sufficiently transitive way on $\mathbb{R}$ (so that an index-two subgroup of $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Homeo}_0(\mathbb{R})$)? [BG98] shows that the structure of these automorphism groups is more intricate when G is one of the generalizations F_n or $F_{n,\infty}$, $n \geq 2$ of F defined in [Bro87]. Indeed, not all the homeomorphisms coming from the automorphism group are piecewise affine in this case.

A similar argument shows that the commensurator group $\mathrm{Comm}(F)$ of F also admits an index-two subgroup with an explicit embedding in $\mathrm{Homeo}_0(\mathbb{R})$ as piecewise dyadically affine homeomorphisms that are eventually periodic and have a countable discrete set of breakpoints [BCR08], but $\mathrm{Comm}(F)$ turns out to be infinitely generated. We do not know if it lies in $\mathcal{C}$.

Chapter 2

Dynamical Tits alternative for groups of almost automorphisms of trees

This chapter corresponds to the paper [GV26].

We prove a dynamical variant of the Tits alternative for the group of almost automorphisms of a locally finite tree $\mathcal{T}$: a group of almost automorphisms of $\mathcal{T}$ either contains a non-abelian free group playing ping-pong on the boundary $\partial\mathcal{T}$, or the action of the group on $\partial\mathcal{T}$ preserves a probability measure. This generalises to all groups of tree almost automorphisms a result of Hurtado-Militon [HM19] for Thompson's group V, with a hopefully simpler proof.

2.1 Context and contributions

The Tits alternative is a celebrated theorem by J. Tits [Tit72] that shows a sharp dichotomy for linear groups over a field of characteristic zero: either they are virtually solvable or they contain a non-abelian free group. A group G is said to satisfy the Tits alternative if for every subgroup H of G, H is virtually solvable or contains a non-abelian free group. This group property has been established for a great deal of countable groups (see the references in [dlH00, Section II B, Complement 42]), usually by applying the Klein ping-pong lemma to exhibit free subgroups.

There are also many countable groups known to fail this alternative, as do many groups of homeomorphisms of compact spaces. For instance, the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of the circle and the group of automorphisms $\mathrm{Aut}(\mathcal{T})$ of a regular tree $\mathcal{T}$ of degree ≥ 3: the former contains Thompson's group F of piecewise dyadically affine homeomorphisms of $[0, 1]$, the latter contains the first Grigorchuk group, and these subgroups are not virtually solvable and do not contain free groups (see [CFP96] and [Gri80], respectively). Nevertheless, these two examples satisfy a dynamical variant of this condition which we formulate as follows.

Definition 2.1.1. *Let X be a compact topological space and G a group of homeomorphisms of X. We say that the action of G on X satisfies* the dynamical Tits alternative *if for every subgroup H of G one of the following holds.*

- *The action of H preserves a regular probability measure on X.*
- *There exists a* ping-pong pair *for the action of H, that is, there exist $g, h \in H$ and disjoint open sets $U_1, U_2, V_1, V_2 \subseteq X$ such that*

$$g(X \setminus U_1) \subseteq V_1 \quad \text{and} \quad h(X \setminus U_2) \subseteq V_2. \tag{2.1.1}$$

This dynamical alternative is a property of a group action, not merely of a group. Nonetheless, if $g, h \in H$ belong to a ping-pong pair, the ping-pong lemma shows that g, h generate a non-abelian free group. Moreover, the conditions in Definition 2.1.1 exclude each other and it suffices to verify them on finitely generated subgroups of G, see the beginning of the proof of Theorem A.

Remark. Previous work [MM23, HM19] involving this notion define the dynamical Tits alternative as a weaker condition, where every subgroup H is required to preserve a probability measure or to contain a non-abelian free group. We prefer our definition since this weaker notion is not a dichotomy, and moreover all known proofs of the alternative yield the stronger condition. For instance, whenever $G \curvearrowright X$ satisfies Definition 2.1.1, a subgroup H of G preserves a probability measure on X if and only if every pair of elements of H preserve a common probability measure on X.

Examples. A first family of examples comes from one-dimensional dynamics: the action on S^1 of the group of homeomorphisms of S^1 satisfies the dynamical Tits alternative by a theorem of G. Margulis [Mar00]. A related example is the Higman-Thompson group V acting on the triadic Cantor set, which also satisfies the alternative by work of Hurtado-Militon [HM19, Theorem 1.3]. A generalization of both statements is given in [MM23, Theorem 1.3], where it is shown that for any compact $K \subseteq \mathbb{R}$, the defining action of the group of locally monotone homeomorphisms of K satisfies the alternative. It is notable that the proof in [MM23] finds sufficiently proximal elements on a group G that does not preserve a measure on K by running a random walk on G, whereas the arguments in [Mar00, HM19] are "deterministic". Groups acting by homeomorphisms on dendrites also satisfy the alternative by work of Duchesne-Monod [DM18, Theorem 1.6].

A second family of examples consists of groups acting on the boundary of Gromov-hyperbolic spaces: a first elementary instance of this class is the action of automorphism group of a locally finite tree $\mathcal{T}$ on its boundary $\partial\mathcal{T}$, as follows easily from the well-known dynamical classification of subgroups of $\mathrm{Aut}(\mathcal{T})$, see [Tit70]. More generally, if (M, d) is a Gromov-hyperbolic and proper metric space such that $\mathrm{Isom}(M, d)$ acts cocompactly on M, then the action of $\mathrm{Isom}(M, d)$ on the Gromov boundary ∂M satisfies the dynamical Tits alternative (see [CCMT15], and also [AS22, Theorem 1.10] for a probabilistic version).

This chapter is concerned with almost automorphism groups of locally finite trees, which are a large family of locally compact and totally disconnected groups that arise as follows:

let $\mathcal{T}$ be a locally finite rooted tree and $\partial\mathcal{T}$ be its space of ends. The group of rooted tree automorphisms $\mathrm{Aut_r}(\mathcal{T})$ acts on $\partial\mathcal{T}$ preserving the so-called visual metric. The group $\mathrm{AAut}(\mathcal{T})$ of almost automorphisms of $\partial\mathcal{T}$ consists of all homeomorphisms of $\partial\mathcal{T}$ that are local homothecies for this metric, that is, that locally rescale it. The natural group topology on $\mathrm{AAut}(\mathcal{T})$ is not the compact-open topology, but the unique group topology such that $\mathrm{Aut}(\mathcal{T})$ is a compact open subgroup of $\mathrm{AAut}(\mathcal{T})$. See Section 2.2 for more precise statements.

If $d \geq 2$, $k \geq 1$ and $\mathcal{T}_{d,k}$ is the rooted tree where the root has k children and all other vertices have d children, then $\mathrm{AAut}(\mathcal{T}_{d,k})$ is known as a Neretin group. These groups are known to be simple [Kap99], compactly presented [LB17] and to contain the Higman-Thompson group $V_{d,k}$ as a dense subgroup. They are the first examples of compactly generated simple groups without lattices [BCGM12] and moreover admit no invariant random subgroups by a result of T. Zheng [Zhe19].

We show that the action of $\mathrm{AAut}(\mathcal{T})$ on $\partial\mathcal{T}$ satisfies the dynamical Tits alternative for any locally finite rooted tree $\mathcal{T}$.

Theorem A. *Let $\mathcal{T}$ be a locally finite rooted tree. The action of* $\mathrm{AAut}(\mathcal{T})$ *on the boundary $\partial\mathcal{T}$ satisfies the dynamical Tits alternative.*

Some interesting groups to which Theorem A applies are the groups V_G considered by V. Nekrashevych in [Nek18], where $G \subseteq \mathrm{Aut_r}(\mathcal{T}_{2,2})$ is a self-similar group. Here V_G is the subgroup of $\mathrm{AAut}(\mathcal{T}_{2,2})$ generated by G and Higman-Thompson's V.

Fix a linear order on $\partial\mathcal{T}$ that is compatible with the tree structure. We denote by $V_\mathcal{T}$ the group of elements of $\mathrm{AAut}(\mathcal{T})$ acting in a locally order-preserving manner. For instance, $V_{\mathcal{T}_{d,k}}$ is the Higman-Thompson group associated to $\mathcal{T}_{d,k}$, and all topological full groups of irreducible infinite one-sided shifts of finite type are naturally subgroups of some $V_\mathcal{T}$, see [Led20, Theorem 3.8]. We call $V_\mathcal{T}$ the *Higman-Thompson group associated to* $\mathcal{T}$, although this name is not standard. The proof of Theorem B below gives a shorter and hopefully more conceptual approach to [HM19, Theorem 1.3] when specialized to the Higman-Thompson group V.

Theorem B. *Let H be a finitely generated subgroup of $V_\mathcal{T}$. Then the action of H on $\partial\mathcal{T}$ has a finite orbit or admits a ping-pong pair.*

We emphasize that Theorem B is not new, since the proof of [HM19, Theorem 1.3] shows that V verifies the (slightly stronger) conclusion of Theorem B and general arguments allow to extend the result from V to any group $V_\mathcal{T}$.

Two important ingredients in both proofs are a characterization of relatively compact subgroups of $\mathrm{AAut}(\mathcal{T})$ by Le Boudec-Wesolek [LBW19] and a description of the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$ by Goffer-Lederle [GL21] (building on work of O. Salazar-Díaz [SD10]).

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2.2 Preliminaries

We give some background on $\mathrm{AAut}(\mathcal{T})$, describe the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$, recall the definition of the Vietoris topology on closed subsets of a space and fix some notation. For more details on this material, see [GL21, LBW19, GL18].

Notation

Given a metric space (X, d), $A \subseteq X$ and $\varepsilon > 0$, we denote $A^\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$. When $\mathcal{T}$ is a rooted tree, we write $\mathrm{Aut}_{\mathrm{r}}(\mathcal{T})$ for the group of tree automorphisms of $\mathcal{T}$ that fix the root.

Almost automorphism groups of trees

Let $\mathcal{T}$ be a locally finite rooted tree with no leaves. We denote its root by v_0, and assume that all edges are directed away from v_0. A *caret* is a subtree of $\mathcal{T}$ consisting of a vertex, its children, and the edges between them. A subtree is *complete* if it is a union of carets, and when $\mathcal{T}_1, \mathcal{T}_2$ are rooted complete subtrees of $\mathcal{T}$ we denote by $\mathcal{T}_2 \setminus \mathcal{T}_1$ the union of all carets in $\mathcal{T}_2$ that are not included in $\mathcal{T}_1$. The set of leaves of a tree $\mathcal{T}$ is denoted by $\mathcal{L}\mathcal{T}$.

Let $\partial\mathcal{T}$ be the *space of ends* of $\mathcal{T}$, that is, the set of (one-sided) infinite directed paths starting at v_0. We equip $\partial\mathcal{T}$ with the topology induced by the *visual metric* defined as $d(\xi, \xi') = 2^{-N(\xi,\xi')}$ for $\xi, \xi' \in \partial\mathcal{T}$ where $N(\xi, \xi') \in \mathbb{N}$ is the smallest integer n such that $\xi_n \neq \xi'_n$. The space $(\partial\mathcal{T}, d)$ is totally disconnected and compact, and its topology has a basis of clopen balls given by the sets $\partial\mathcal{T}_v = \{\xi \in \partial\mathcal{T} : v \text{ is in } \xi\}$ where $v \in V(\mathcal{T})$.

An *almost automorphism* of $\mathcal{T}$ is a homeomorphism g of $\partial\mathcal{T}$ such that there exists a partition of $\partial\mathcal{T}$ into clopen balls $D_1, \ldots, D_n$ and positive numbers $\lambda_1, \ldots, \lambda_n$ such that $d(g(y), g(z)) = \lambda_j d(y, z)$ for all $y, z \in D_j$. Such a partition is said to be *admissible* for g. Another way of viewing an almost automorphism is the following: take $\mathcal{T}_1, \mathcal{T}_2$ finite subtrees of $\mathcal{T}$ with root v_0, so $\mathcal{T} \setminus \mathcal{T}_1$, $\mathcal{T} \setminus \mathcal{T}_2$ are naturally rooted forests. Then any isomorphism $\overline{g}\colon \mathcal{T} \setminus \mathcal{T}_1 \to \mathcal{T} \setminus \mathcal{T}_2$ of rooted forests determines a $g \in \mathrm{AAut}(\mathcal{T})$, and conversely any almost automorphism arises in this manner, although not uniquely so.

Call an almost automorphism $g \in \mathrm{AAut}(\mathcal{T})$ *elliptic* if there exists a partition $\mathcal{P}$ of $\partial\mathcal{T}$ into clopen balls that is admissible for g and g-invariant, that is, such that $g\mathcal{P} = \mathcal{P}$. The following theorem is stated for Neretin groups in [LBW19], but the proof for any locally finite tree $\mathcal{T}$ is the same word by word.

Proposition 2.2.1 ([LBW19, Corollary 3.6]). *Let H be a finitely generated subgroup of $\mathrm{AAut}(\mathcal{T})$. The following are equivalent.*

- *H is relatively compact in $\mathrm{AAut}(\mathcal{T})$.*

- *Every element of H is elliptic.*

- *There is a partition $\mathcal{P}$ of $\partial\mathcal{T}$ into clopen balls such that $\mathcal{P}$ is admissible for every $h \in H$.*

Moreover, if these conditions hold and $\mathcal{Q}$ is any partition of $\partial\mathcal{T}$ into clopen balls such that $\mathcal{Q}$ is admissible for every element of some generating set of H, then $\mathcal{P}$ can be chosen to be finer than $\mathcal{Q}$.

Fix a family of total linear orders $\{<_v : v \in V(\mathcal{T})\}$ indexed by the vertices of $\mathcal{T}$, where $<_v$ is an order on the children of $v \in V(\mathcal{T})$. This family induces a linear order $<$ on $\partial\mathcal{T}$ by declaring $\xi < \xi'$ if $\xi, \xi' \in \partial\mathcal{T}$ and $\xi_n <_{\xi_{n-1}} \xi'_n$, where $n = N(\xi, \xi') \in \mathbb{N}$. We call $g \in \mathrm{AAut}(\mathcal{T})$ a *Higman-Thompson element* if there exists an admissible partition $\mathcal{P}$ for g such that $g\big|_D$ is order-preserving for all $D \in \mathcal{P}$. The subset of Higman-Thompson elements of $\mathrm{AAut}(\mathcal{T})$ is a group that we denote by $V_\mathcal{T}$. We omit $\{<_v : v \in V(\mathcal{T})\}$ from the definition of $V_\mathcal{T}$ to keep the notation uncluttered, but for non-regular trees $\mathcal{T}$ the group $V_\mathcal{T}$ may depend on the choice of these orders.

Since the elliptic elements of $V_\mathcal{T}$ are exactly the elements of finite order in $V_\mathcal{T}$, as in [LBW19] Proposition 2.2.1 yields the following result.

Corollary 2.2.2 ([LBW19, Corollary 3.7])**.** *Any finitely generated subgroup of $V_\mathcal{T}$ composed entirely of elliptic elements is finite.*

Dynamics of almost automorphisms

We will make use of a description of the dynamics of individual elements of $\mathrm{AAut}(\mathcal{T})$, which is one of the subjects of [GL21]. Again, the proofs are given for $\mathcal{T} = \mathcal{T}_{d,k}$ but a careful reading shows that all arguments hold in the general case.

We need some definitions, following [GL21]: a *tree pair* is a tuple $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ are finite complete trees with root v_0 and $\kappa\colon \mathcal{L}\mathcal{T}_1 \to \mathcal{L}\mathcal{T}_2$ is a bijection between the leaves of $\mathcal{T}_1$ and the leaves of $\mathcal{T}_2$. We say that a tree pair $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is *associated* to an element $g \in \mathrm{AAut}(\mathcal{T})$ if g arises from an isomorphism of rooted forests $\overline{g}\colon \mathcal{T}\backslash\mathcal{T}_1 \to \mathcal{T}\backslash\mathcal{T}_2$ such that $\kappa = \overline{g}\big|_{\mathcal{L}\mathcal{T}_1}$. In this case $\{\partial\mathcal{T}_v\}_{v\in\mathcal{L}\mathcal{T}_1}$ is an admissible partition for g. We will only consider tree pairs that are associated to some almost automorphism of $\mathcal{T}$. All tree pairs are associated to almost automorphisms when $\mathcal{T} = \mathcal{T}_{d,k}$ but this is not true in general since the connected components of $\mathcal{T} \setminus \mathcal{T}_1$ and $\mathcal{T} \setminus \mathcal{T}_2$ need not be isomorphic.

Consider an orbit $\mathcal{O} = \{u_0, \ldots, u_n\} \subseteq \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ of the partial action of κ on $\mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$, that is, $\mathcal{O}$ is such that $u_0, \ldots, u_{n-1} \in \mathcal{L}\mathcal{T}_1$, $u_1, \ldots, u_n \in \mathcal{L}\mathcal{T}_2$, $\kappa(u_j) = u_{j+1}$ for all $j = 0, \ldots, n-1$ and either

- $u_0 \notin \mathcal{L}\mathcal{T}_2$ and $u_n \notin \mathcal{L}\mathcal{T}_1$, or

- $u_0 \in \mathcal{L}\mathcal{T}_2$, $u_n \in \mathcal{L}\mathcal{T}_1$ and $\kappa(u_n) = u_0$.

The orbit $\mathcal{O}$ is said to be

- an *attracting chain* if u_n is a descendant of u_0 in $\mathcal{T}$, in which case u_n is called the *attractor* of the orbit,
- a *repelling chain* if u_0 is a descendant of u_n in $\mathcal{T}$, in which case u_0 is called the *repeller* of the orbit,
- a *periodic chain* if $\kappa(u_n) = u_0$, and
- a *wandering chain* if $u_0 \notin \mathcal{T}_2$ (that is, u_0 is a descendant of some leaf in $\mathcal{L}\mathcal{T}_2$) and $u_n \notin \mathcal{T}_1$ (that is, u_n is a descendant of some leaf in $\mathcal{L}\mathcal{T}_1$).

These options are mutually exclusive. We say that $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is a *revealing tree pair* if each connected component of $\mathcal{T}_1 \setminus \mathcal{T}_2$ contains a repeller and each connected component of $\mathcal{T}_2 \setminus \mathcal{T}_1$ contains an attractor. In this case, if $C \subseteq \mathcal{T}_1 \setminus \mathcal{T}_2$ is a connected component containing a repeller u_0, the orbit $\{u_0, \dots, u_n\}$ is such that u_n is the root of C and u_0 is the unique repeller in C. Similarly, if $C \subseteq \mathcal{T}_2 \setminus \mathcal{T}_1$ is a connected component containing an attractor u_n, the orbit $\{u_0, \dots, u_n\}$ is such that u_0 is the root of C and u_n is the unique attractor in C. Revealing tree pairs were introduced by O. Salazar-Díaz in [SD10] to describe the dynamics of individual elements of Thompson's V.

If $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ is a revealing tree pair, then every orbit $\mathcal{O}$ is an attracting, repelling, periodic or wandering chain: assume that $\mathcal{O}$ is not periodic, so it must end in an element $u_n \in \mathcal{L}\mathcal{T}_2 \setminus \mathcal{L}\mathcal{T}_1$ and begin in an element $u_0 \in \mathcal{L}\mathcal{T}_1 \setminus \mathcal{L}\mathcal{T}_2$. If $u_0 \in \mathcal{T}_2$, then u_0 is the root of its component of $\mathcal{T}_2 \setminus \mathcal{T}_1$ and u_n must be the attractor in this component. If $u_n \in \mathcal{T}_1$, then u_n is the root of its component in $\mathcal{T}_1 \setminus \mathcal{T}_2$ and u_0 must be the repeller in this component. If neither happen, then $\mathcal{O}$ is wandering.

Theorem 2.2.3 ([GL21, Lemma 2.17]). *Every element of* $\mathrm{AAut}(\mathcal{T})$ *is associated to some revealing tree pair.*

As a consequence we deduce the following corollary, the proof of which uses ideas present in [GL21, Section 3.1] (compare [HM19, Lemma 5.5] for Higman-Thompson's group V).

Corollary 2.2.4. *If* $g \in \mathrm{AAut}(\mathcal{T})$ *there is a partition* $\partial\mathcal{T} = U_g \sqcup V_g$ *into* g*-invariant clopen subsets such that* U_g *is equal to the open subset of* $\xi \in \partial\mathcal{T}$ *that admit a neighborhood* $U \subseteq \partial\mathcal{T}$ *such that some positive power of* $g\big|_U$ *is an isometry onto* U*. Moreover, the following properties are verified.*

- *There is a positive power of* $g\big|_{U_g}$ *that is an isometry (for the visual metric of* $\partial\mathcal{T}$*).*
- *There are finitely many* g*-periodic points in* V_g*, which we denote* $\mathrm{Per}_{\mathrm{hyp}}(g)$*.*
- *There is a partition* $\mathrm{Per}_{\mathrm{hyp}}(g) = \mathrm{Per}_{\mathrm{rep}}(g) \sqcup \mathrm{Per}_{\mathrm{att}}(g)$ *into* repelling *and* attracting *periodic points, such that for every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *so that for all* $k \geq N$ *we have*

$$g^k(V_g \setminus \mathrm{Per}_{\mathrm{rep}}(g)^\varepsilon) \subseteq \mathrm{Per}_{\mathrm{att}}(g)^\varepsilon \quad \text{and} \quad g^{-k}(V_g \setminus \mathrm{Per}_{\mathrm{att}}(g)^\varepsilon) \subseteq \mathrm{Per}_{\mathrm{rep}}(g)^\varepsilon. \tag{2.2.1}$$

Proof. Let $(\kappa, \mathcal{T}_1, \mathcal{T}_2)$ be a revealing pair associated to g. Write $\widetilde{U}_g = \bigsqcup \partial\mathcal{T}_u$ where the union is over all vertices $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ that are in periodic chains, and set $\widetilde{V}_g = \partial\mathcal{T} \setminus \widetilde{U}_g$. The sets $\widetilde{U}_g$ and $\widetilde{V}_g$ are g-invariant and clopen, and if $\partial\mathcal{T}_u \subseteq \widetilde{U}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$, there exists $n \in \mathbb{N}$ such that $g^n(\partial\mathcal{T}_u) = \partial\mathcal{T}_u$ and $g^n\big|_{\partial\mathcal{T}_u}$ is an isometry. By taking appropriate powers of g we see that there is a $k \in \mathbb{N}$ such that $g^k\big|_{\widetilde{U}_g}$ is an isometry.

Now let $\partial\mathcal{T}_u \subseteq \widetilde{V}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in an attracting chain $\{u_0, \ldots, u_n\}$. We have that $g^n(\partial\mathcal{T}_{u_0}) \subsetneq \partial\mathcal{T}_{u_0}$ and $g^n\big|_{\partial\mathcal{T}_{u_0}}$ is a contraction, so in particular $g^n(\partial\mathcal{T}_u) \subsetneq \partial\mathcal{T}_u$ and $g^n\big|_{\partial\mathcal{T}_u}$ is a contraction. Thus there exists a unique g^n-fixed point $\xi_u \in \partial\mathcal{T}_u$, and which must also verify $\bigcap_{k\geq 0} g^{kn}(\partial\mathcal{T}_u) = \{\xi_u\}$. Set

$$\mathrm{Per}_{\mathrm{att}}(g) = \{\xi_u : u \text{ is in an attracting chain}\}.$$

If $\partial\mathcal{T}_u \subseteq \widetilde{V}_g$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in a repelling chain instead, the argument of the previous paragraph applied to g^{-1} shows that for some $n \in \mathbb{N}$ there exists a unique g^n-fixed point $\xi_u \in \partial\mathcal{T}_u$ such that $\bigcap_{k\geq 0} g^{-kn}(\partial\mathcal{T}_u) = \{\xi_u\}$. Set

$$\mathrm{Per}_{\mathrm{rep}}(g) = \{\xi_u : u \text{ is in a repelling chain}\}.$$

The sets $\mathrm{Per}_{\mathrm{att}}(g)$ and $\mathrm{Per}_{\mathrm{rep}}(g)$ are disjoint and finite. Moreover, their union gives all g-periodic points in $\widetilde{V}_g$ since any $\xi \in \partial\mathcal{T}_u$ where $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ is in a wandering chain cannot be a g-periodic point: indeed, if the orbit $\{u_0, \ldots, u_n\}$ is wandering, then the connected component of $u_n \in \mathcal{L}\mathcal{T}_2 \setminus \mathcal{T}_1$ in $\mathcal{T}_2 \setminus \mathcal{T}_1$ has a root $u_{\mathrm{a}} \in \mathcal{L}\mathcal{T}_1$ which must be the first element of an attracting orbit. Thus

$$d\left(g^{n-k+j}\left(\partial\mathcal{T}_{u_k}\right), g^j\left(\xi_{u_{\mathrm{a}}}\right)\right) \xrightarrow{j\to\infty} 0$$

for every $k = 0, \ldots, n$. In the same way, the root $u_{\mathrm{r}} \in \mathcal{L}\mathcal{T}_2$ of the connected component of $u_0 \in \mathcal{L}\mathcal{T}_1 \setminus \mathcal{T}_2$ in $\mathcal{T}_1 \setminus \mathcal{T}_2$ is the last element of a repelling orbit, so

$$d\left(g^{-k-j}\left(\partial\mathcal{T}_{u_k}\right), g^{-j}\left(\xi_{u_{\mathrm{r}}}\right)\right) \xrightarrow{j\to\infty} 0$$

for every $k = 0, \ldots, n$. Since $\xi_{u_{\mathrm{r}}} \neq \xi_{u_{\mathrm{a}}}$, there are no periodic points in any $\partial\mathcal{T}_{u_k}$ for $k = 0, \ldots, n$.

To prove that $\widetilde{V}_g$, $\mathrm{Per}_{\mathrm{att}}(g)$, $\mathrm{Per}_{\mathrm{rep}}(g)$ verify the last item of the corollary, by symmetry it suffices to prove the first statement in (2.2.1). To do this, take a clopen $U \subsetneq \widetilde{V}_g \setminus \mathrm{Per}_{\mathrm{rep}}(g)$ and $\varepsilon > 0$, and let $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$. If u is in an attracting or wandering chain, it is clear that there is an $N \in \mathbb{N}$ such that $g^k(U \cap \partial\mathcal{T}_u) \subseteq \mathrm{Per}_{\mathrm{att}}(g)^{\varepsilon}$ for all $k \geq N$. If u is in a repelling orbit $\{u_0, \ldots, u_n\}$, then the equality

$$\bigcap_{k\geq 0} g^{-kn}(\partial\mathcal{T}_u) \cap (U \cap \partial\mathcal{T}_u) = \varnothing$$

shows that there is a $l \in \mathbb{N}$ such that $g^{-kn}(\partial\mathcal{T}_u) \cap (U \cap \partial\mathcal{T}_u) = \varnothing$ for all $k \geq l$. The sets $g^r(\partial\mathcal{T}_u)$, $r = 0, \ldots, n-1$ are pairwise disjoint, and we conclude that $\partial\mathcal{T}_u \cap g^k(U \cap \partial\mathcal{T}_u) = \varnothing$ for all $k \geq nl$. Upon replacing U by $g^r(U)$ for some sufficiently big $r \in \mathbb{N}$ we may assume that $U \cap \partial\mathcal{T}_u$ is empty for all $u \in \mathcal{L}\mathcal{T}_1 \cup \mathcal{L}\mathcal{T}_2$ in a repelling orbit. Hence (2.2.1) holds for a sufficiently large $N \in \mathbb{N}$.

Up to now, $\widetilde{U}_g$, $\widetilde{V}_g$ satisfy all the required properties except that $\widetilde{V}_g$ could still contain some points that admit a neighborhood $U \subseteq \partial\mathcal{T}$ such that some positive power of $g\big|_U$ is an isometry onto U. Call the set of such points $\mathcal{I} \subseteq \widetilde{V}_g$. All elements in $\widetilde{V}_g \setminus (\mathrm{Per}_{\mathrm{rep}}(g) \sqcup \mathrm{Per}_{\mathrm{att}}(g))$ admit a neighborhood U such that $g^j(U) \cap U = \varnothing$ for all $j \neq 0$, so $\mathcal{I} \subseteq \mathrm{Per}_{\mathrm{rep}}(g) \sqcup \mathrm{Per}_{\mathrm{att}}(g)$ is finite. Thus setting $U_g = \widetilde{U}_g \sqcup \mathcal{I}$, $V_g = \widetilde{V}_g \setminus \mathcal{I}$ ensures that all properties in the statement of the corollary are verified. □

In contrast with the trees $\mathcal{T}_{d,k}$, the boundary of an arbitrary locally finite tree $\mathcal{T}$ may contain isolated points and the three items in the previous corollary do not specify U_g, V_g uniquely. The definition of U_g in the statement of the corollary gives a canonical definition in any case.

Vietoris topology

Given a topological space X, we denote by 2^X the space of closed subsets of X equipped with the *Vietoris topology*, defined as the topology with subbasis

$$\{K \in 2^X : K \cap U \neq \varnothing,\, K \cap V = \varnothing\}$$

where U, V range over all open subsets of X. If X is compact metrizable then 2^X is also compact metrizable [Eng89, Section 4.5], and in this case any action by homeomorphisms of a countable group on X induces naturally an action by homeomorphisms on 2^X [Eng89, Section 3.12].

2.3 Proofs

We first gather some useful lemmas.

Lemma 2.3.1 (Neumann's lemma, [Neu54, Lemma 4.1]). *Let G be a group acting on a set X and assume that the action has no finite orbits. Then for every pair of finite subsets A, B of X there exists an element $g \in G$ such that $g(A) \cap B$ is empty.*

Lemma 2.3.2 ([MM23, Proposition 1.16]). *Let G be a group acting by homeomorphisms on a compact metric space (X, d). Assume that*

i. there exists a positive integer p such that for any $\varepsilon > 0$ there exist non-empty finite sets $A, B \subseteq X$ of cardinality at most p with $g(X \setminus A^\varepsilon) \subseteq B^\varepsilon$ for some $g \in G$, and

ii. there are no finite G-orbits.

Then the action of G on X has a ping-pong pair.

Proof. Notice that condition (i) is equivalent to the existence of finite sets A, B of X that work for *any* $\varepsilon > 0$: a pair (A, B) of non-empty subsets of X of cardinality at most p is called a *contraction pair* if for every neighborhood U, V of A, B respectively there exists $g \in G$ such that $g(X \setminus U) \subseteq V$. Then condition (i) and the compactness of X imply that there exists a contraction pair.

Moreover, if (A, B) is a contraction pair and $u, v \in G$ then $(u(A), v(B))$ is a contraction pair too. Thus, by (ii) and Lemma 2.3.1, our contraction pair (A, B) can be taken such that A and B are disjoint.

Using (ii) and Lemma 2.3.1 again we deduce that there exist two contraction pairs (A_1, B_1), (A_2, B_2) where the A_1, A_2, B_1, B_2 are pairwise disjoint. If $U_i, V_i, i = 1, 2$ are pairwise disjoint neighborhoods of $A_i, B_i, i = 1, 2$ respectively, we can find $g_1, g_2 \in G$ such that $g_i(X \setminus U_i) \subseteq V_i$ for $i = 1, 2$. These constitute a ping-pong pair. □

Lemma 2.3.3. *Let G be a compact topological group and $g \in G$. For every neighborhood U of the identity there exists a strictly increasing sequence $(n_j)_{j\geq 0} \subseteq \mathbb{N}$ such that $g^{n_j} \in U$ for all $j \in \mathbb{N}$.*

Proof. Let μ be the normalized Haar measure on G, so that μ is a probability measure of complete support on G and left multiplication by g preserves μ. Let $V \subseteq U$ be an open neighborhood of the identity such that $V \cdot V^{-1} \subseteq U$. Since $\mu(V) > 0$, Poincaré's recurrence theorem implies that there is a strictly increasing sequence $(n_j)_{j\geq 0} \subseteq \mathbb{N}$ such that $\mu(V \cap g^{n_j} V) > 0$ for all $j \in \mathbb{N}$. In particular $g^{n_j} \in V \cdot V^{-1} \subseteq U$ for all $j \in \mathbb{N}$. □

The core of the proof of Theorem A is the following statement, which uses ideas from [LBMB22, Proposition 5.2] and implies the extreme contraction properties required by Lemma 2.3.2. For an almost automorphism $g \in \mathrm{AAut}(\mathcal{T})$ we use the same notation as Corollary 2.2.4, so we write $U_g \subseteq \partial\mathcal{T}$ for the stable set of g and $\mathrm{Per}_{\mathrm{hyp}}(g)$, $\mathrm{Per}_{\mathrm{rep}}(g)$ for the hyperbolic and repelling points of g respectively.

Recall that we denote by $2^{\partial\mathcal{T}}$ the compact metrizable space of all closed subsets of $\partial\mathcal{T}$ equipped with the Vietoris topology.

Proposition 2.3.4. *Let H be a subgroup of $\mathrm{AAut}(\mathcal{T})$ and assume that $\bigcap_{h\in H} U_h$ is empty. Then there exists a finite set $B \subseteq \partial\mathcal{T}$ such that for every $\varepsilon > 0$ there is a $h \in H$ with $h(\partial\mathcal{T} \setminus B^\varepsilon) \subseteq B^\varepsilon$.*

Proof. By compactness there exist elements $h_1, \ldots, h_k \in H$ such that $\bigcap_{j=1}^k U_{h_j}$ is empty. By taking powers of the h_j we can assume that every $h_j\big|_{U_j}$ is an isometry. Set $B = \bigcup_{j=1}^k \mathrm{Per}_{\mathrm{hyp}}(h_j)$ and $C_1 = \partial\mathcal{T} \setminus B^\varepsilon$.

Now h_1 restricted to U_{h_1} is an isometry. Since isometry groups of compact spaces are compact for the compact-open topology, Lemma 2.3.3 and a diagonal argument shows that there exists a strictly increasing sequence $(n_j)_{j\geq 0} \subseteq \mathbb{N}$ such that

$$\sup_{x\in U_{h_1}} d\left(h_1^{n_j}(x), x\right) \leq \frac{1}{j}$$

for all $j \in \mathbb{N}$.

For any closed $D \subseteq \partial\mathcal{T}$ denote by $\overline{\mathrm{Orb}_H(D)}$ the closure of the H-orbit of D in $2^{\partial\mathcal{T}}$. By taking a limit point of $(h_1^{n_j}(C_1))_{j\geq 0}$ in $2^{\partial\mathcal{T}}$ we obtain a closed $C_2 \in \overline{\mathrm{Orb}_H(C_1)}$ with

$$C_2 \subseteq (C_1 \cap U_{h_1}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_1).$$

Lemma 2.3.3 applied to h_2 gives again the existence of a sequence $(n'_j)_{j\geq 0} \subseteq \mathbb{N}$ such that

$$\sup_{x\in U_{h_2}} d\left(h_2^{n'_j}(x), x\right) \leq \frac{1}{j}$$

for all $j \in \mathbb{N}$. The finite set $\mathrm{Per}_{\mathrm{rep}}(h_2)$ can only intersect C_2 in $\mathrm{Per}_{\mathrm{hyp}}(h_1)$, so by taking again a limit point of $(h_2^{n'_j}(C_2))_{j\geq 0}$ we obtain a closed $C_3 \in \overline{\mathrm{Orb}_H(C_2)} \subseteq \overline{\mathrm{Orb}_H(C_1)}$ with

$$C_3 \subseteq (C_2 \cap U_{h_2}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_2) \subseteq (C_1 \cap U_{h_1} \cap U_{h_2}) \sqcup (\mathrm{Per}_{\mathrm{hyp}}(h_1) \cup \mathrm{Per}_{\mathrm{hyp}}(h_2)).$$

Notice that, again, the finite set $\mathrm{Per}_{\mathrm{rep}}(h_3)$ can only intersect C_3 in $\mathrm{Per}_{\mathrm{hyp}}(h_1) \cup \mathrm{Per}_{\mathrm{hyp}}(h_2)$.

By iterating this argument inductively we produce, for every $j = 2, \ldots, k+1$, a closed subset $C_j \in \overline{\mathrm{Orb}_H(C_{j-1})} \subseteq \overline{\mathrm{Orb}_H(C_1)}$ such that

$$C_j \subseteq (C_{j-1} \cap U_{h_{j-1}}) \sqcup \mathrm{Per}_{\mathrm{hyp}}(h_{j-1}) \subseteq (C_1 \cap U_{h_1} \cap \cdots \cap U_{h_{j-1}}) \sqcup \left(\bigcup_{i=1}^{j-1} \mathrm{Per}_{\mathrm{hyp}}(h_i)\right).$$

In particular $C_{k+1} \subseteq \bigcup_{i=1}^{k} \mathrm{Per}_{\mathrm{hyp}}(h_i) = B$, and we are done. □

Proof of Theorem A. Take a subgroup H of $\mathrm{AAut}(\mathcal{T})$ and suppose that H does not preserve a probability measure on $\partial\mathcal{T}$. If every finitely generated subgroup of H preserves a probability measure on $\partial\mathcal{T}$, the compactness of $\mathrm{Prob}(\partial\mathcal{T})$ equipped with the weak-$*$ topology implies that H preserves a probability measure on $\partial\mathcal{T}$. We may assume then that H is finitely generated.

Suppose that the set $U_H = \bigcap_{h\in H} U_h$ is non-empty. Since every U_h is dynamically defined (see the definition of U_h in Corollary 2.2.4), we have that U_H is H-invariant. The closed set U_H may be written as $\partial\mathcal{T}'$ for some rooted subtree $\mathcal{T}' \subseteq \mathcal{T}$, and we equip U_H with the restriction of the visual metric of $\partial\mathcal{T}$, which coincides with the visual metric of $\partial\mathcal{T}'$. Every element of H has a proper power that acts as an isometry on $\partial\mathcal{T}'$, so the action of H on $\partial\mathcal{T}'$ is by elliptic almost automorphisms of $\mathcal{T}'$. Since H is finitely generated, Proposition 2.2.1 shows that the image of H in $\mathrm{AAut}(\mathcal{T}')$ is relatively compact. Thus H preserves a measure on $\partial\mathcal{T}' \subseteq \partial\mathcal{T}$, a contradiction. We conclude that U_H is empty, and in this case Proposition 2.3.4 and Lemma 2.3.2 imply that there exists a ping-pong pair for the action of H. □

Remark. The proof of Theorem A shows a slightly stronger statement, namely that for any finitely generated subgroup H of $\mathrm{AAut}(\mathcal{T})$, either H contains a ping-pong pair or there exists a non-empty closed set $C \subseteq \partial\mathcal{T}$ such that the action of H on C is equicontinuous.

Recall that $V_{\mathcal{T}}$ denotes the Higman-Thompson group associated to $\mathcal{T}$.

Proof of Theorem B. Fix a linear order on $\partial\mathcal{T}$ as in the definition of $V_{\mathcal{T}}$. Take H a finitely generated subgroup of $V_{\mathcal{T}}$ and suppose that H acts without finite orbits on $\partial\mathcal{T}$. If $U_H = \bigcap_{h\in H} U_h$ is non-empty, consider again a rooted subtree $\mathcal{T}' \subseteq \mathcal{T}$ such that $U_H = \partial\mathcal{T}'$, so H acts on U_H by elliptic Higman-Thompson elements of $\mathrm{AAut}(\mathcal{T}')$ for the induced order on $\partial\mathcal{T}'$. Corollary 2.2.2 shows that the image of H in $\mathrm{AAut}(\mathcal{T}')$ is finite, and thus there exists a finite H-orbit in U_H. This is a contradiction, hence U_H is empty. Again Proposition 2.3.4 and Lemma 2.3.2 show that there exists a ping-pong pair for the action of H. □

2.4 Open questions

We finish with some hopefully tractable open questions.

Question. The whole homeomorphism group of the Cantor space does not verify the dynamical Tits alternative.[1] Nevertheless, Grigorchuk-Nekrashevych-Sushchansky introduce in [GNS00] the group $\mathcal{R}$ of homeomorphisms of Cantor space defined by asynchronous transducers, which is known [BB17] to contain the higher-dimensional Brin-Thompson groups from [Bri04]. On the other hand, the automorphism group of a Higman-Thompson group $V_{d,k}$ coincides with the subgroup of bi-synchronizing transducers inside $\mathcal{R}$ [BCM+24]. Does the natural action of any of these groups on the Cantor set satisfy the dynamical Tits alternative?

Question. Let G be a group of homeomorphisms of a compact topological space X such that its groupoid of germs is hyperbolic in the sense of V. Nekrashevych, see [Nek15]. Does the action of G on X satisfy the dynamical Tits alternative? Theorem A shows that this is true for the groups V_G, whose groupoid of germs is hyperbolic whenever G is a contracting self-similar group.

[1]Take any non-amenable countable group G without non-abelian free subgroups. Since G is non-amenable, there is a subshift $X \subseteq \{0,1\}^G$ with no invariant probability measures [FKSV25, Theorem 5.13]. X cannot be countable, so it contains a G-invariant Cantor subset Y. The action $G \curvearrowright Y$ does not satisfy the alternative.

Chapter 3

Probabilistic Tits alternative for circle diffeomorphisms

This chapter corresponds to the paper [GV25].

To state the main result, let μ_1, μ_2 be probability measures on $\mathrm{Diff}^1_0(S^1)$ satisfying a suitable moment condition and such that their supports genererate discrete groups acting proximally on S^1. Let $(f^n_\omega)_{n\geq 0}, (f^n_{\omega'})_{n\geq 0}$ be two independent realizations of the random walk driven by μ_1, μ_2 respectively. We show that almost surely there is an $N \in \mathbb{N}$ such that for all $n \geq N$ the elements $f^n_\omega, f^n_{\omega'}$ generate a non-abelian free group. The proof is inspired by the strategy by R. Aoun for linear groups. A weaker (and easier) statement holds for measures supported on $\mathrm{Homeo}_0(S^1)$ with no moment conditions.

3.1 Context and contributions

The Tits alternative is a celebrated theorem by J. Tits which asserts that finitely generated linear groups are either virtually solvable or contain a non-abelian free group [Tit72]. This alternative fails for groups of homeomorphisms of the circle, but a weaker alternative (sometimes called a *dynamical Tits alternative*, see [MM23]) still holds.

Theorem 3.1.1 (G. Margulis [Mar00], see also V. Antonov [Ant84])**.** *Let G be a subgroup of* $\mathrm{Homeo}(S^1)$. *Then either*

i. G is elementary*, that is, the action of G preserves a probability measure on S^1, or*

ii. G contains a ping-pong pair*, that is, two elements $f, g \in G$ such that there are pairwise disjoint open subsets U_1, U_2, V_1, V_2 of S^1 with $f(S^1 \setminus U_1) \subseteq V_1$, $g(S^1 \setminus U_2) \subseteq V_2$.*

The previous options are mutually exclusive and, by the ping-pong lemma, condition (ii) implies that f, g generate a non-abelian free group in G.

We are interested in how generic these two elements are. As a first approximation, there is a dense G_δ subset W of $\mathrm{Homeo}(S^1) \times \mathrm{Homeo}(S^1)$ such that any pair of elements in W generate a non-abelian free group [Ghy01, Proposition 4.5] (see also [Tri14, Theorem 6.9]). Our viewpoint will be probabilistic instead of topological, inspired by the following result of R. Aoun for linear groups. We first fix some notation: a probability measure μ on a group G is said to be *non-degenerate* if the semigroup generated by its support is all G. Given non-degenerate probability measures μ_1, μ_2 on groups G_1, G_2 we let $(\Omega_i, \mathbb{P}_i)$, $i = 1, 2$ be the probability space $(G_i^{\mathbb{N}}, \mu_i^{\otimes \mathbb{N}})$, and we write $\omega = (f_{\omega_n})_{n\geq 0}$ for an element of Ω_1 and $\omega' = (f_{\omega'_n})_{n\geq 0}$ for an element of Ω_2. Also, denote f_ω^n for the right random walk $f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$ at time $n \in \mathbb{N}$.

Theorem 3.1.2 (R. Aoun [Aou11, Aou13])**.** *Let G be a real algebraic linear group that is semisimple and with no compact factors, and let G_1, G_2 be Zariski-dense subgroups of G. If μ_1, μ_2 are non-degenerate probability measures on G_1, G_2 respectively with finite exponential moment, then there exists $\rho \in (0, 1)$ such that*

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[(\omega, \omega') \in \Omega_1 \times \Omega_2 \textit{ such that } f_\omega^n, f_{\omega'}^n \textit{ are a ping-pong pair}\right] \geq 1 - \rho^n$$

for all sufficiently large $n \in \mathbb{N}$.

In the previous statement, two elements $f, g \in \mathrm{GL}_d(\mathbb{R}), d \in \mathbb{N}$ are said to be a *ping-pong pair* if the conditions in (ii) hold for their natural action on the projective space $\mathrm{P}\mathbb{R}^d$, and the measure μ is said to have *exponential moment* if $\int_G \|g\|^\delta \, \mathrm{d}\mu(g)$ is finite for some $\delta > 0$ (here $\|\cdot\|$ is any norm on $d \times d$ matrices).

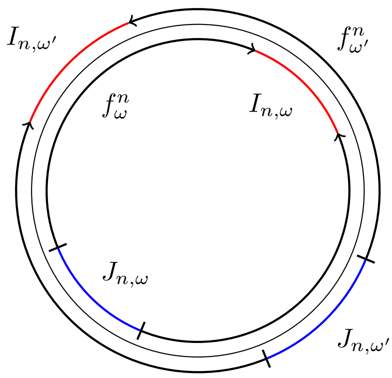

Figure 3.1

When specialized to $\mathrm{PSL}_2(\mathbb{R})$ acting on S^1, the proof shows that the situation depicted in Figure 3.1 occurs with probability converging to 1 exponentially fast in $n \in \mathbb{N}$. That is, there exist disjoint intervals $I_{n,\omega}, I_{n,\omega'}, J_{n,\omega}, J_{n.\omega'} \subseteq S^1$ that testify that $f_\omega^n, f_{\omega'}^n$ are a ping-pong pair. The intervals $I_{n,\omega}, I_{n,\omega'}$ can be taken centered around $f_\omega^n(x), f_{\omega'}^n(y)$ where $x, y \in S^1$ are arbitrary and fixed beforehand. The intervals $J_{n,\omega}, J_{n,\omega'}$ converge as n increases to the *repellers* $\sigma(\omega), \sigma(\omega')$ of the random walks $f_\omega^n, f_{\omega'}^n$ (see the next section for the definition of σ). To control the probabilities that they intersect, their diameters decrease to 0 exponentially fast in n.

The main result of this chapter shows that this situation remains typical for a pair of independent random walks on countable subgroups of $\mathrm{Diff}^1_0(S^1)$, the group of orientation-preserving diffeomorphisms of S^1, provided the action of the subgroups on S^1 is proximal. This condition is almost always fulfilled for a group acting on S^1 admitting no invariant measures on S^1, see Theorem 3.2.2 below for a precise statement. For a function $\phi\colon S^1 \to \mathbb{R}$, set

$$|\phi|_{\mathrm{Lip}} = \sup_{x\neq y\in S^1} \frac{|\phi(x)-\phi(y)|}{d(x,y)}.$$

Theorem C. *Let G_1, G_2 be countable subgroups of $\mathrm{Diff}^1_0(S^1)$ such that the actions of G_1 and of G_2 on S^1 are proximal. Let μ_1, μ_2 be non-degenerate probability measures on G_1, G_2 respectively such that there exists $\delta > 0$ so that the integral*

$$\int_{G_i} \max\left\{|g|_{\mathrm{Lip}}, \left|g^{-1}\right|_{\mathrm{Lip}}\right\}^{\delta} \mathrm{d}\mu(g)$$

is finite for $i = 1, 2$.

Then there exists $\rho \in (0,1)$ such that

$$\mathbb{P}_1 \otimes \mathbb{P}_2\left[(\omega,\omega') \in \Omega_1 \times \Omega_2 \textit{ such that } f^n_\omega, f^n_{\omega'} \textit{ are a ping-pong pair}\right] \geq 1-\rho^n$$

for all sufficiently large $n \in \mathbb{N}$.

As with Theorem 3.1.2, the Borel-Cantelli lemma immediately implies the following.

Corollary D. *Let μ_1, μ_2 be probability measures on $\mathrm{Diff}^1_0(S^1)$ satisfying the same assumptions as in Theorem C. For $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every (ω, ω') there exists $N \in \mathbb{N}$ such that $f^n_\omega, f^n_{\omega'}$ generate a non-abelian free group for all $n \geq N$.*

The conclusion of Theorem C is known to be true in other settings: if M is a proper hyperbolic space such that $\mathrm{Isom}(M)$ acts cocompactly on M and μ is a measure on $\mathrm{Isom}(M)$ generating a non-elementary group, then this is [AS22, Theorem 1.10]. The case of non-elementary hyperbolic groups acting on their Gromov boundary and finitely supported μ was treated previously in [GMO10].

We do not know if the statement in Corollary D is true for groups of bi-Lipschitz homeomorphisms of S^1.[1] To put this in perspective, we show that a weakening of Corollary D is true for groups of homeomorphisms of S^1, even without moment assumptions on the measures μ_i and relaxing the proximality assumption on the G_i to the absence of G_i-invariant probability measures on S^1. It is an application of results in [DKN07], but has not appeared previously in the literature up to our knowledge. Denote by $\mathrm{Homeo}_0(S^1)$ the group of orientation-preserving homeomorphisms of S^1.

Theorem E. *Let G_1, G_2 be countable subgroups of $\mathrm{Homeo}_0(S^1)$ such that the actions of G_1 and G_2 on S^1 do not admit any invariant probability measures, and let μ_1, μ_2 be non-degenerate*

[1] The relevance of the bi-Lipschitz condition comes from the fact that any countable subgroup of $\mathrm{Homeo}_0(S^1)$ is conjugated to a group of bi-Lipschitz homeomorphisms, see [DKN07, Théorème D].

probability measures on G_1, G_2 respectively. Then for $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every $(\omega, \omega') \in \Omega_1 \times \Omega_2$, the set of $n \in \mathbb{N}$ such that f^n_ω, $f^n_{\omega'}$ are a ping-pong pair has density 1, that is,

$$\lim_{N\to\infty} \frac{1}{N} |\{0 \le n \le N : f^n_\omega,\, f^n_{\omega'} \text{ are a ping-pong pair}\}| = 1.$$

The proof of Theorem E requires only the tools developped in [DKN07, Appendice] and general statements on contracting random dynamical systems from [Mal17]. In contrast, the proof of Theorem C requires more ingredients. For instance, to apply the strategy of [Aou11] in this context it is essential to know that exponential contractions occur in mean and that the stationary measure is Hölder continuous (see Theorems 3.4.2 and 3.2.6 below respectively). The first condition has been already proven in different situations in the literature by Gelfert-Salcedo [GS23], Gorodetski-Kleptsyn [GK21] and Barrientos-Malicet [BM24], all of which require (at least) that μ be supported on $\mathrm{Diff}^1_0(S^1)$. The second one is a very general theorem by Gorodetski-Kleptsyn-Monakov [GKM22]. One important difference with the linear setting lies in the dynamics of individual elements of $\mathrm{Homeo}_0(S^1)$: very "contracting" homeomorphisms of the circle do not have a canonically defined repeller or attractor. Proposition 3.4.4 below deals with this issue.

Remark. Theorem C and Corollary D are also true when the supports of the probability measures μ_i only generate *semigroup* (as opposed to a groups), with the same proofs. However, the proof of Theorem E uses the existence of maps $\pi_i \colon S^1 \to S^1$ intertwining the given actions of the G_i with minimal proximal actions. When G_i is only a semigroup with no invariant probability measure such a map may not exist, see [KKO18, Section 9.5].

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3.2 Preliminaries

We review some basic theory on random dynamical systems and groups acting on the circle, and introduce some notation. For more details on this material, see [Ghy01, DKN07, Mal17].

Notation

Given a metric space (X, d), $A \subseteq X$ and $\varepsilon > 0$, we write $A^\varepsilon = \{x \in X : d(x, A) \le \varepsilon\}$. We also write 1_A for the indicator function on A and δ_x for the Dirac mass on a point $x \in X$. We will denote by d the metric on S^1 coming from a fixed identification $S^1 = \mathbb{R}/\mathbb{Z}$, so that $\mathrm{diam}(S^1) = 1/2$. Probability measures on S^1 in this chapter are always assumed to be Borel.

Random dynamical systems

For us, a *random dynamical system* $(G, \mu) \curvearrowright X$ (or a *random walk on* X) is the data of a countable group G acting by homeomorphisms on a compact metric space (X, d) and of a probability measure μ on G. We always assume that μ is *non-degenerate*, that is, that the semigroup generated by the support of μ is G. We remark that this notation does not coincide with that of [Mal17], where a measure on a semigroup is said to be non-degenerate if its support generates the whole semigroup. Since we deal only with groups no confusion will arise.

Denote by $(\Omega, \mathbb{P})$ the probability space $(G^{\mathbb{N}}, \mu^{\otimes\mathbb{N}})$ and set $f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_0}$ when $n \in \mathbb{N}$, $\omega = (f_{\omega_k})_{k\geq 0} \in \Omega$. A *$\mu$-stationary measure* is a Borel probability measure on X such that $\mu * \nu = \nu$, where

$$\mu * \nu(A) = \sum_{g\in G} \mu(g)\nu(g^{-1}A)$$

for all Borel $A \subseteq X$.

We say that $(G, \mu) \curvearrowright X$ is *locally contracting* if for all $x \in X$, $\mathbb{P}$-almost surely there exists a neighborhood $B \subseteq X$ of x such that $\operatorname{diam}(f_\omega^n(B)) \xrightarrow[n\to\infty]{} 0$.

Proposition 3.2.1 ([Mal17, Propositions 4.8 and 4.9])**.** *Suppose $(G, \mu) \curvearrowright X$ is locally contracting. Then there are finitely many ergodic μ-stationary measures $\nu_1, \ldots, \nu_d$, and their supports are exactly the minimal G-invariant sets in X.*

Moreover, for every $x \in X$ and $\mathbb{P}$-almost every $\omega \in \Omega$ there exists a unique index $i = i(\omega, x) \in \{1, \ldots, d\}$ such that $f_\omega^n(x)$ equidistributes towards ν_i*, that is*

$$\frac{1}{N}\sum_{n=0}^{N} \delta_{f_\omega^n(x)} \xrightarrow[N\to\infty]{} \nu_i$$

in the weak $*$-topology.

Groups acting on the circle

We say that a group action $G \curvearrowright S^1$ is *proximal* if for every strict subinterval $I \subsetneq S^1$ and $\varepsilon > 0$ there exists $g \in G$ such that $\operatorname{diam}(g(I)) < \varepsilon$. The action is said to be *locally proximal* if every $x \in S^1$ is the endpoint of an interval $I \subseteq S^1$ such that for all $\varepsilon > 0$ there exists $g \in G$ with $\operatorname{diam}(g(I)) < \varepsilon$. In this context, we say that a group action $G \curvearrowright^{\phi} S^1$ is *semiconjugate* to $G \curvearrowright^{\psi} S^1$ if there exists a continuous surjection $\pi\colon S^1 \to S^1$ such that $\psi(g) \circ \pi = \pi \circ \phi(g)$ for all $g \in G$, and such that π is locally non-decreasing and has degree one (this means that any lift of π to $\widetilde{\pi}\colon \mathbb{R} \to \mathbb{R}$ is non-decreasing and satisfies $\widetilde{\pi}(x+1) = \widetilde{\pi}(x) + 1$ for all $x \in \mathbb{R}$). When such a π does not necessarily have degree 1, we call it a *factor map*.

The following theorem is essentially equivalent to Theorem 3.1.1 from the introduction: in cases (ii.b) and (ii.c) below G always contains a free group, and there exists an invariant probability measure for G in cases (i) (a mean of Dirac masses on a finite orbit on S^1) and (ii.a) (the image of Lebesgue measure under a conjugacy to a group of rotations).

Theorem 3.2.2 (see [Ghy01])**.** *Consider an action $G \curvearrowright^{\phi} S^1$ by orientation-preserving homeomorphisms. Then exactly one of the following options is satisfied.*

i. There exists a finite orbit.

ii. There exists a unique closed minimal set Λ, which is either S^1 or a Cantor set. In the latter case, by collapsing the countably many connected components of $S^1 \setminus \Lambda$ we can semiconjugate ϕ to a minimal group action $G \curvearrowright S^1$.

Moreover, in the minimal case a further distinction exists: either

ii.a the action is free and thus conjugated to an action by rotations, or

ii.b the action is proximal, or

ii.c the action is locally proximal and not proximal, and there exists $d \in \mathbb{N}_+$, $d \geq 2$ and a continuous d-to-one covering $\pi\colon S^1 \to S^1$ that intertwines ϕ with a proximal action.

Thus whenever $G \curvearrowright^{\phi} S^1$ does not preserve any probability measure on S^1, there exists $d \in \mathbb{N}_+$ and a factor map $\pi\colon S^1 \to S^1$ that is d-to-one on the minimal set of G (except for a countable number of points), and that intertwines ϕ with a proximal and minimal action. We will call this integer d the *degree of proximality* of ϕ, but this notation is not standard.

Random walks on S^1

In this subsection, fix a countable group G and a non-degenerate probability measure μ on G. The random walk on S^1 defined by a proximal group action $G \curvearrowright^{\phi} S^1$ has been well studied.

Theorem 3.2.3 ([DKN07, Appendice])**.** *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1.*

i. There exists a unique μ-stationary probability measure ν on S^1, which is atomless and is supported on the minimal set of G.

ii. If the action of G on S^1 is proximal, there exists a random variable $\omega \in \Omega \mapsto \sigma(\omega) \in S^1$ such that for $\mathbb{P}$-almost every ω we have

$$(f_\omega^n)^{-1}\nu \xrightarrow[n\to\infty]{} \delta_{\sigma(\omega)}$$

in the weak-$*$ topology.

We call the random variable $\sigma(\omega)$ from the previous theorem the *repeller* of the random walk $(f_\omega^n)_{n\geq 0}$. Its distribution is the unique $\overline{\mu}$-stationary measure on S^1 where $\overline{\mu} \in \mathrm{Prob}(G)$ is defined on $g \in G$ as $\mu(g^{-1})$, and is thus non-atomic.

Notice that the measure $(f_\omega^n)^{-1}\nu$ is given by $\nu(f_\omega^n(I))$ on every interval $I \subseteq S^1$, so the statement in (ii) says that $f_\omega^n(I)$ is contracted into a ν-null set unless I contains $\sigma(\omega)$, in which case it is expanded to the whole circle. As a consequence, for all $x, y \in S^1$ we have $\mathbb{P}$-almost surely that x and y are not $\sigma(\omega)$ since the law of $\sigma(\omega)$ is non-atomic, and hence $\lim_{n\to\infty} d(f_\omega^n(x), f_\omega^n(y)) = 0$. This conclusion is the subject of [KN04] (see also [Ant84]), and justifies the fact that we will use $f_\omega^n(0)$ (or $f_\omega^n(x)$ for some non-random $x \in S^1$) as an “attractor” for f_ω^n in the proofs below.

When $G \curvearrowright S^1$ is not necessarily proximal a similar statement holds, and even more is true: the rate of contraction of $f_\omega^n(I)$ when $\sigma(\omega) \notin I$ is exponential.

Theorem 3.2.4. *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1. Let $d \in \mathbb{N}_+$ be the degree of proximality of $G \curvearrowright S^1$.*

There exist measurable functions $\sigma_1, \ldots, \sigma_d \colon \Omega \to S^1$ such that the following hold.

i. [Mal17, Theorem D] There exists $\lambda > 0$ such that for $\mathbb{P}$-almost every ω and every closed interval $I \subseteq S^1 \setminus \{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$ we have

$$\operatorname{diam}(f_\omega^n(I)) \leq e^{-\lambda n}$$

for all sufficiently large $n \in \mathbb{N}$.

ii. [Mal17, Proposition 4.3] $\mathbb{P}$-almost surely, the set $\{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$ has size d.

The random set $\{\sigma_1, \ldots, \sigma_d\}$ from the previous theorem is called the *repelling set* of the random walk $(f_\omega^n)_{n\geq 0}$. In this setting, let $\pi \colon S^1 \to S^1$ be a factor map to a minimal and proximal action and $\Lambda \subseteq S^1$ be the minimal set of $G \curvearrowright S^1$. Define $E \subseteq S^1$ as the countable set of images of connected components of $S^1 \setminus \Lambda$, so for all $x \in S^1 \setminus E$, the fiber $\pi^{-1}(x)$ has size d. Denote by $\sigma(\omega)$ the repeller of the random walk in the image of π. If $\sigma(\omega) \in S^1 \setminus E$ (which happens $\mathbb{P}$-almost surely, since the distribution of σ is non-atomic), then

$$\pi^{-1}(\sigma(\omega)) = \{\sigma_1(\omega), \ldots . \sigma_d(\omega)\}$$

by the defining properties of $\sigma(\omega)$ and $\{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$. We record this as a proposition.

Proposition 3.2.5. *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1. Denote by $\pi \colon S^1 \to S^1$ a factor map to a minimal and proximal action, and by $\omega \mapsto \sigma(\omega)$ the repeller of the random walk induced in the image of π.*

Then $\mathbb{P}$-almost surely the repelling set $F(\omega) = \{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$ of $(f_\omega^n)_{n\geq 0}$ satisfies the equality $\pi^{-1}(\sigma(\omega)) = F(\omega)$.

We finish with the Hölder regularity of the unique μ-stationary measure of a proximal random dynamical system $(G, \mu) \curvearrowright S^1$. The original statement in [GKM22, Theorem 2.3] is written for a subgroup G of $\mathrm{Diff}^1(M)$ for any compact smooth manifold M, but [GKM22, Remark 2.10] shows that differentiability of the maps in G is not essential: what is truly needed is that all maps of G be bi-Lipschitz.

Theorem 3.2.6 ([GKM22, Theorem 2.3])**.** *Consider an action $G \curvearrowright S^1$ by orientation-preserving diffeomorphisms of class C^1 with no invariant probability measure on S^1, and assume that for some $\delta > 0$ the integral*

$$\int_G \max\left\{|g|_{\mathrm{Lip}}, \left|g^{-1}\right|_{\mathrm{Lip}}\right\}^\delta \mathrm{d}\mu(g)$$

is finite.

*Then there exist $C, \alpha > 0$ such that any μ-stationary probability measure ν on S^1 is (C, α)-*Hölder continuous*, that is, $\nu(B(x, r)) \leq Cr^\alpha$ for all $x \in S^1$ and $r > 0$.*

3.3 Probabilistic Tits alternative in $\mathrm{Homeo}_0(S^1)$

This section proves Theorem E. All the relevant tools were already introduced in the previous section.

Proof of Theorem E. Fix μ_1, μ_2 two non-degenerate probability measures on countable subgroups G_1, G_2 of $\mathrm{Homeo}_0(S^1)$ that do not preserve any probability measure on S^1. For $i = 1, 2$, let

- ν_i be the unique μ_i-stationary measure on S^1 and $\Lambda_i \subseteq S^1$ the minimal set of G_i,
- $d_i \in \mathbb{N}_+$ be the degree of proximality of $G_i \curvearrowright S^1$, and
- $\pi_i \colon S^1 \to S^1$ be a factor map of $G_i \curvearrowright S^1$ to a minimal proximal action of G_i such that π_i is d_i-to-one ν_i-almost everywhere.

Recall that the degree of proximality of $G_i \curvearrowright S^1$ is the unique integer such that the map π_i exists, see the discussion after Theorem 3.2.2.

For $\omega \in \Omega_1$ we write

- $(g^n_\omega)_{n\geq 0}$ for the random walk driven by μ_1 acting on the image of π_1 and $\sigma(\omega) \in S^1$ for its repelling point, and
- $F(\omega) \subseteq S^1$ for the repelling set of the random walk $(f^n_\omega)_{n\geq 0}$.

Recall also that π_1 intertwines the action $G_1 \curvearrowright S^1$ with a minimal proximal action of G_1, so the random walk $(g^n_\omega)_{n\geq 0}$ verifies the conclusion of Theorem 3.2.3, (ii). When $\omega' \in \Omega_2$ we denote by $g^n_{\omega'}$, $\sigma(\omega')$ and $F(\omega')$ the same objects associated to μ_2.

Fix once and for all $x \in S^1$, $y \in S^1$ such that $\pi_1^{-1}(x)$ and $\pi_2^{-1}(y)$ have size d_1, d_2 respectively and are disjoint.

Claim. *The following properties are true for $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every $(\omega, \omega') \in \Omega_1 \times \Omega_2$.*

i. The sequence $\{(f^n_\omega(a), f^n_{\omega'}(b))\}_{n\geq 0} \subseteq S^1 \times S^1$ equidistributes with respect to $\nu_1 \otimes \nu_2$ for every $a \in \pi_1^{-1}(x)$ and $b \in \pi_2^{-1}(y)$.

ii. The equalities $F(\omega) = \pi_1^{-1}(\sigma(\omega))$ and $F(\omega') = \pi_2^{-1}(\sigma(\omega'))$ hold.

iii. The sets $F(\omega)$, $F(\omega')$, $\pi_1^{-1}(x)$ and $\pi_2^{-1}(y)$ are pairwise disjoint.

Proof of the claim. (i) The random dynamical system $(G_1 \times G_2, \mu_1 \otimes \mu_2) \curvearrowright S^1 \times S^1$ is locally contracting since $(G_1, \mu_1) \curvearrowright S^1$, $(G_2, \mu_2) \curvearrowright S^1$ are locally contracting. Moreover, for any pair $(x, y) \in S^1 \times S^1$ the orbit $\mathrm{Orb}_{G_1\times G_2}((x, y))$ accumulates on $\Lambda_1 \times \Lambda_2$, so $\Lambda_1 \times \Lambda_2$ is the unique $G_1 \times G_2$-minimal set and Proposition 3.2.1 shows that $S^1 \times S^1$ has a unique $\mu_1 \otimes \mu_2$-stationary measure, namely $\nu_1 \otimes \nu_2$. Again Proposition 3.2.1 gives equidistribution $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost surely.
(ii) This is Proposition 3.2.5.
(iii) By independence it suffices to show that $\mathbb{P}_1 [z \in F(\omega)] = \mathbb{P}_2 [z \in F(\omega')] = 0$ for any fixed $z \in S^1$, but this follows from $\mathbb{P}_1 [z \in F(\omega)] = \mathbb{P}_1 [\pi_1(z) = \sigma(\omega)]$ and the fact that the distribution of $\sigma(\omega)$ is non-atomic. □

We will assume in what follows that the pair $(\omega, \omega') \in \Omega_1 \times \Omega_2$ satisfies the previous properties. Fix $\varepsilon > 0$ and pick $\delta > 0$ such that any interval $I \subseteq S^1$ with $|I| \leq \delta$ has $\nu_1(I), \nu_2(I) \leq \varepsilon$ and also $\nu_1 \otimes \nu_2(\triangle^\delta) \leq \varepsilon$ where $\triangle \subseteq S^1 \times S^1$ is the diagonal (here $S^1 \times S^1$ is equipped with the ℓ^∞-metric). Choose $\chi = \chi(\omega, \omega') \in (0, \delta/2)$ such that $F(\omega)^\chi$ and $F(\omega')^\chi$ are disjoint.

Suppose that $I \subseteq S^1$ has diameter at most 2χ and consider $a \in \pi_1^{-1}(x)$, $b \in \pi_2^{-1}(y)$. Equidistribution implies that the quantities

$$\limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N : f_\omega^n(a) \in I\}|, \quad \limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N : f_{\omega'}^n(b) \in I\}|$$
$$\text{and} \qquad \limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N : (f_\omega^n(a), f_{\omega'}^n(b)) \in \triangle^\chi\}|$$

are all smaller than ε. By considering all the combinations in which the intervals in $f_\omega^n(\pi_1^{-1}(x))^\chi$, $f_{\omega'}^n(\pi_2^{-1}(y))^\chi$, $F(\omega)^\chi$ and $F(\omega')^\chi$ can intersect, we conclude that

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N : & f_\omega^n(\pi_1^{-1}(x))^\chi,\ f_{\omega'}^n(\pi_2^{-1}(y))^\chi, \\ & F(\omega)^\chi \text{ and } F(\omega')^\chi \text{ are not pairwise disjoint}\}|\end{aligned} \tag{3.3.1}$$

is at most $(d_1^2 + d_2^2 + 3d_1d_2)\varepsilon$. More explicitly, (3.3.1) is at most

$$\begin{aligned}\sum_{\substack{a\in\pi_1^{-1}(x), b\in\pi_2^{-1}(y)\\ f\in F(\omega), f'\in F(\omega')}} & \limsup_{N\to\infty} \frac{1}{N} \Big(|\{0 \leq n < N : d(f_\omega^n(a), f_{\omega'}^n(b)) \leq 2\chi\}| \\ & + |\{0 \leq n < N : d(f_\omega^n(a), f) \leq 2\chi\}| + |\{0 \leq n < N : d(f_{\omega'}^n(b), f') \leq 2\chi\}| \\ & + |\{0 \leq n < N : d(f_\omega^n(a), f') \leq 2\chi\}| + |\{0 \leq n < N : d(f_{\omega'}^n(b), f) \leq 2\chi\}| \Big) \\ & \leq d_1d_2\varepsilon + d_1^2\varepsilon + d_2^2\varepsilon + d_1d_2\varepsilon + d_2d_2\varepsilon,\end{aligned}$$

as desired.

Take $\bar\chi > 0$ such that for $i = 1, 2$, the connected components of $\pi_i^{-1}(I)$ have diameter at most χ if $I \subseteq S^1$ has diameter at most $\bar\chi$. By Theorem 3.2.3, $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost surely we can find $n_0 = n_0(\omega, \omega') \in \mathbb{N}$ such that for all $n \geq n_0$ the inclusions

$$g_\omega^n(S^1 \setminus \sigma(\omega)^{\bar\chi}) \subseteq g_\omega^n(x)^{\bar\chi} \quad \text{and} \quad g_{\omega'}^n(S^1 \setminus \sigma(\omega')^{\bar\chi}) \subseteq g_{\omega'}^n(y)^{\bar\chi}$$

hold, so $F(\omega) = \pi_1^{-1}(\sigma(\omega))$, $F(\omega') = \pi_2^{-1}(\sigma(\omega'))$ shows that

$$f_\omega^n(S^1 \setminus F(\omega)^\chi) \subseteq f_\omega^n(\pi_1^{-1}(x))^\chi \quad \text{and} \quad f_{\omega'}^n(S^1 \setminus F(\omega')^\chi) \subseteq f_{\omega'}^n(\pi_2^{-1}(y))^\chi. \tag{3.3.2}$$

The inclusions (3.3.2) imply that every n in the set

$$\mathcal{N} = \{n \geq n_0 : f_\omega^n(\pi_1^{-1}(x))^\chi,\ f_{\omega'}^n(\pi_2^{-1}(y))^\chi,\ F(\omega)^\chi \text{ and } F(\omega')^\chi \text{ are pairwise disjoint}\}$$

is such that $f_\omega^n, f_{\omega'}^n$ are a ping-pong pair. Thus

$$\begin{aligned}\varliminf_{N\to\infty} \frac{1}{N} |\{0 \leq n < N : f_\omega^n,\ f_{\omega'}^n \text{ are a ping-pong pair}\}| & \geq \varliminf_{N\to\infty} \frac{1}{N} |\mathcal{N} \cap [0, N]| \\ & \geq 1 - (d_1^2 + d_2^2 + 3d_1d_2)\varepsilon,\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary the conclusion follows. □

3.4 Probabilistic Tits alternative in $\mathrm{Diff}_0^1(S^1)$

The purpose of this section is to prove Theorem C. Subsection 3.4.1 gives some preliminary inequalities for groups of diffeomorphisms acting proximally on S^1 (Theorem 3.4.2 and Proposition 3.4.3), and Subsection 3.4.2 gives the proof of Theorem C. In this section G is a countable subgroup of $\mathrm{Diff}_0^1(S^1)$ that acts proximally on S^1 and μ is a non-degenerate probability measure on G such that

$$\text{there exists } \delta > 0 \text{ so that } \int_G \max\left\{|g|_{\mathrm{Lip}}, |g^{-1}|_{\mathrm{Lip}}\right\}^\delta \mathrm{d}\mu(g) \text{ is finite.} \tag{M}$$

3.4.1 Preliminary statements

We now state and prove Theorem 3.4.2, which gives (uniform) exponential contractions in mean in our context. It is a variation on similar statements that have appeared independently in [GK21, Proposition 4.18] and [GS23, Theorem 1.3], assuming that μ has finite support in $\mathrm{Diff}_0^1(S^1)$. The proof follows [GK21, Proposition 4.18] closely, along with additional input from [BM24, Proposition 4.5]. This is the only point in the proof where we use that μ is supported in $\mathrm{Diff}_0^1(S^1)$, namely to obtain inequality (3.4.1) below.

Theorem 3.4.1 ([BM24, Proposition 4.5])**.** *If μ satisfies the condition* (M)*, there exist constants $r, \lambda > 0$, $s_0 \in (0,1]$ and $k \in \mathbb{N}_+$ such that*

$$\mathbb{E}\left[d(f_\omega^{k_1}(x), f_\omega^{k_1}(y))^s\right] \le e^{-\lambda} d(x,y)^s \tag{3.4.1}$$

for all $x, y \in S^1$ such that $d(x,y) \le r$ and all $s \in (0, s_0]$.

Theorem 3.4.2. *There exist $\lambda_+ > 0$, $s \in (0,1]$ and $N \in \mathbb{N}$ such that for all $n \ge N$ we have*

$$\sup_{x \neq y \in S^1} \mathbb{E}\left[\frac{d(f_\omega^n(x), f_\omega^n(y))^s}{d(x,y)^s}\right] \le e^{-\lambda_+ n}.$$

In particular, for all $n \ge N$ we have

$$\sup_{x,y \in S^1} \mathbb{E}\left[d(f_\omega^n(x), f_\omega^n(y))\right] \le e^{-\lambda_+ n}$$

Proof. Take $r, \lambda > 0$, $s_0 \in (0,1]$ and $k_1 = k \in \mathbb{N}_+$ given by Theorem 3.4.1, so that (3.4.1) holds for all $x, y \in S^1$ with $d(x,y) \le r$ and all $s \in (0, s_0]$.

Claim. *For every $\varepsilon_1, \varepsilon_2 > 0$ there exists $k_2 \in \mathbb{N}_+$ such that*

$$\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < \varepsilon_1\right] > 1 - \varepsilon_2$$

for all $x, y \in S^1$.

Proof of the claim. This is [GK21, Lemma 4.23], but we give the proof for completeness.

Let $l \in \mathbb{N}_+$ be large enough so that the points $x_j = j/l \in S^1$, $0 \le j \le l-1$ satisfy

$$\mathbb{P}\left[\sigma(\omega) \in (x_j, x_{j+1})\right] \le \varepsilon_2/4$$

for each j. When $0 \le j \le l-1$ denote by I_j the open interval with endpoints x_j, x_{j+1} and length $1-1/l$. By the defining property of $\sigma(\omega)$, we may choose $k_2 \in \mathbb{N}$ large enough so that for every $0 \le j \le l-1$, we have

$$\mathbb{P}\left[\mathrm{diam}(f_\omega^{k_2}(I_j)) \le \varepsilon_1 \mid \sigma(\omega) \in (x_j, x_{j+1})\right] \ge 1 - \varepsilon_2/2.$$

If $x, y \in S^1$, then $x, y \in I_j$ for all indices $0 \le j \le l-1$ except at most two values j_1, j_2. Thus $\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < \varepsilon_1\right]$ is bounded below by

$$\sum_{\substack{j=0 \\ j \ne j_1, j_2}}^{l-1} \mathbb{P}\left[\mathrm{diam}(f_\omega^{k_2}(I_j)) \le \varepsilon_1 \mid \sigma(\omega) \in (x_j, x_{j+1})\right] \mathbb{P}\left[\sigma(\omega) \in (x_j, x_{j+1})\right] \ge (1-\varepsilon_2/2)(1-2\varepsilon_2/4)$$

$$\ge 1 - \varepsilon_2. \qquad \square$$

Fix $s \in (0, s_0]$ and find $k_2 \in \mathbb{N}_+$ such that

$$\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < r/4^{1/s}\right] > 1 - r^s/4$$

for all $x, y \in S^1$. If we take $x, y \in S^1$ such that $d(x, y) \ge r$, then by conditioning on whether $d(f_\omega^{k_2}(x), f_\omega^{k_2}(y))$ is smaller or larger than $r/4$ we obtain

$$\begin{aligned} \mathbb{E}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y))^s\right] &\le \mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) > r/4^{1/s}\right] + \left(\frac{r}{4^{1/s}}\right)^s \mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < r/4^{1/s}\right] \\ &\le \frac{r^s}{4} + \frac{r^s}{4} \le \frac{1}{2} d(x, y)^s. \end{aligned} \tag{3.4.2}$$

For every $k \in \mathbb{N}_+$ and $x, y \in S^1$, define a random variable $K_k(\omega) \in \mathbb{N}_+$ (which depends on x, y) as follows: if $d(x, y) \le r$ (resp. $d(x, y) > r$) apply k_1 (resp. k_2) random iterations of $\omega = (f_{\omega_n})_{n \ge 0}$ to the pair x, y. Repeat the process on the pair $f_\omega^{k_1}(x), f_\omega^{k_1}(y)$ (resp. $f_\omega^{k_2}(x), f_\omega^{k_2}(y)$) applying iterations of $(f_{\omega_{n+k_1}})_{n \ge 0}$ (resp. $(f_{\omega_{n+k_2}})_{n \ge 0}$) until the total number of iterations exceeds k for the first time. By definition we have $k \le K_k \le k + \max\{k_1, k_2\}$.

Claim. *The inequality*

$$\mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^s\right] \le \max\left\{\left(\frac{1}{2}\right)^{\frac{k}{k_2}}, e^{-\frac{k}{k_1}\lambda}\right\} d(x, y)^s \tag{3.4.3}$$

holds.

Proof. Define $r_1(\omega)$ as the (random) number of times where k_1 elements of the $(f_{\omega_i})_{i \ge 0}$ were applied in the definition of $K_k(\omega)$ and define $r_2(\omega)$ similarly, so $r_1(\omega)k_1 + r_2(\omega)k_2 = K_k(\omega)$. The Markov property and inequalities (3.4.1), (3.4.2) show that

$$\mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^s\right] \le \mathbb{E}\left[\left(\frac{1}{2}\right)^{r_2(\omega)} e^{-\lambda r_1(\omega)}\right] d(x, y). \tag{3.4.4}$$

Set $r = r_1(\omega)k_1 + r_2(\omega)k_2$ and $c_1 = e^{-\lambda}, c_2 = 1/2$, so

$$r_1(\omega)\ln c_1 + r_2(\omega)\ln c_2 = r\left(\frac{r_1(\omega)k_1}{r}\frac{\ln c_1}{k_1} + \frac{r_2(\omega)k_2}{r}\frac{\ln c_2}{k_2}\right) \le r \max\left\{\frac{\ln c_1}{k_1}, \frac{\ln c_2}{k_2}\right\}. \tag{3.4.5}$$

The right-hand side of (3.4.5) is at most $k \max\left\{\frac{\ln c_1}{k_1}, \frac{\ln c_2}{k_2}\right\}$ since the c_i belong to $(0,1)$ and $r \geq k$. By exponentiating we obtain

$$\left(\frac{1}{2}\right)^{r_2(\omega)} e^{-\lambda r_1(\omega)} \leq \max\left\{\left(\frac{1}{2}\right)^{\frac{k}{k_2}}, e^{-\frac{k}{k_1}\lambda}\right\},$$

which together with (3.4.4) proves the desired conclusion. □

The f_{ω_i} are independent and distributed along μ, and hence

$$\begin{aligned}\mathbb{E}\left[\left|\left(f_{\omega_{K_k}} \circ \cdots \circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s}\right] &\leq \mathbb{E}\left[\left|f^{-1}_{\omega_{K_k}}\right|_{\mathrm{Lip}}^{s} \cdots \left|f^{-1}_{\omega_k}\right|_{\mathrm{Lip}}^{s}\right] \\ &\leq \mathbb{E}\left[\left|f^{-1}_{\omega_{k+\max\{k_1,k_2\}}}\right|_{\mathrm{Lip}}^{s} \cdots \left|f^{-1}_{\omega_k}\right|_{\mathrm{Lip}}^{s}\right] \\ &= \int_G \left|g^{-1}\right|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}} \mathrm{d}\mu(g). \end{aligned} \tag{3.4.6}$$

We deduce that

$$\begin{aligned}\mathbb{E}\left[d(f^k_\omega(x), f^k_\omega(y))^{s/2}\right] &\leq \mathbb{E}\left[\left|\left(f_{\omega_{K_k}} \circ \cdots \circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s/2} d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^{s/2}\right] \\ &\leq \mathbb{E}\left[\left|\left(f_{\omega_{K_k}} \circ \cdots \circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s}\right]^{1/2} \mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^{s}\right]^{1/2} \\ &\leq \left(\int_G \left|g^{-1}\right|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}} \mathrm{d}\mu(g)\right)^{1/2} \max\left\{\left(\frac{1}{2}\right)^{\frac{k}{2k_2}}, e^{-\frac{k}{2k_1}\lambda}\right\} d(x,y)^{s/2},\end{aligned}$$

where we have used (3.4.3) and (3.4.6) in the last inequality. If $s \leq \delta/\max\{k_1,k_2\}$, where $\delta > 0$ is provided by the condition (M), the term $\int_G |g^{-1}|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}} \mathrm{d}\mu(g)$ is also finite. But the term $\max\left\{\left(\frac{1}{2}\right)^{\frac{k}{2k_2}}, e^{-\frac{k}{2k_1}\lambda}\right\}$ converges to 0 as $k \to \infty$, and hence there exists $k \in \mathbb{N}_+$ and $\lambda > 0$ such that

$$\mathbb{E}\left[d(f^k_\omega(x), f^k_\omega(y))^s\right] \leq e^{-\lambda} d(x,y)^s$$

for all $x, y \in S^1$ and $0 < s \leq \min\{s_0/2, \delta/(2k_1), \delta/(2k_2)\}$. By the Markov property, for all $n \in \mathbb{N}$ we have

$$\mathbb{E}\left[d(f^n_\omega(x), f^n_\omega(y))^s\right] \leq e^{-\lambda\lfloor n/k\rfloor} d(x,y)^s \leq e^{\lambda} e^{-\lambda n/k} d(x,y)^s.$$

The conclusion follows by setting $\lambda_+ = \lambda/(2k)$ and choosing $N \in \mathbb{N}$ so that $e^{\lambda}e^{-\lambda N/(2k)} < 1$. □

Remark. The previous theorem says that $(G,\mu) \curvearrowright S^1$ is *μ-contracting*, according to the terminology of Benoist-Quint in [BQ16, Section 11.1]. As a consequence, all of the limit laws available for cocycles in this setting (that is, the central limit theorem, the law of the iterated logarithm and large deviations estimates, see [BQ16, Section 12.1]) hold in this setting. In particular the Lyapunov cocycle $(g,x) \mapsto \log g'(x)$ satisfies these limit laws. This recovers [GS24, Theorem 1.14], for instance. However, [BQ16, Theorem 12.1] requires the cocycle to be Lipschitz with integrable Lipschitz constant, but the proofs go through without relevant changes if the Lipschitz condition is replaced by a τ-Hölder one.

Denote by $(\overline{f}_\omega^n)_{n\geq 0}$ the *left (or inverse) random walk* $\overline{f}_\omega^n = f_{\omega_0} \circ f_{\omega_1} \circ \cdots f_{\omega_n}$. Define the random variable $T(\omega) \in S^1$ as the repeller of the random walk $\left(f_{\omega_n}^{-1} \circ f_{\omega_{n-1}}^{-1} \circ \cdots \circ f_{\omega_0}^{-1}\right)_{n\geq 0}$. The following is an analogue of [Aou11, Theorem 4.16].

Proposition 3.4.3. *Let* $\lambda_+ > 0$ *be the constant provided by Theorem 3.4.2. There exist constants* $\lambda_- > 0$, $N \in \mathbb{N}$ *such that*

$$\sup_{x\in S^1} \mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), \sigma(\omega)\right)\right] \leq e^{-\lambda_- n} \tag{3.4.7}$$

and

$$\sup_{x\in S^1} \mathbb{E}\left[d\left(\overline{f}_\omega^n(x), T(\omega)\right)\right] \leq e^{-\lambda_+ n} \tag{3.4.8}$$

hold for $n \geq N$*. Moreover, there exist* $C_-, \alpha_- > 0$ *such that the distribution of* T *is* (C_-, α_-)*-Hölder continuous.*

Proof. By applying (3.4.7) to the random walk on $\mathrm{Diff}_0^1(S^1)$ driven by $\overline{\mu}$ where $\overline{\mu}(g) = \mu(g^{-1})$ for all $g \in \mathrm{Diff}_0^1(S^1)$ we conclude that (3.4.8) holds. The Hölder continuity of the distribution of T also follows from Theorem 3.2.6 since this distribution is μ-stationary. It suffices then to prove (3.4.7).

Take $n, k \in \mathbb{N}$ with $0 < n < k$ and fix $x, y \in S^1$. We have that

$$\mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), \sigma(\omega)\right)\right] \leq \mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), (f_\omega^k)^{-1}(y)\right)\right] + \mathbb{E}\left[d\left((f_\omega^k)^{-1}(y), \sigma(\omega)\right)\right].$$

Theorem 3.4.2 applied to the random walk driven by $\overline{\mu}$ gives $\lambda_- > 0$ such that

$$\sup_{u,v\in S^1} \mathbb{E}\left[d\left(f_{\omega_n}^{-1} \circ \cdots \circ f_{\omega_0}^{-1}(u), f_{\omega_n}^{-1} \circ \cdots \circ f_{\omega_0}^{-1}(v)\right)\right] \leq e^{-\lambda_- n}$$

for all sufficiently large $n \in \mathbb{N}$. In particular we deduce that

$$\begin{aligned}
\mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), (f_\omega^k)^{-1}(y)\right)\right] &= \int \mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), (f_\omega^n)^{-1} \circ \gamma^{-1}(y)\right)\right] \mathrm{d}\mu^{*(k-n)}(\gamma) \\
&\leq \sup_{u,v\in S^1} \mathbb{E}\left[d\left((f_\omega^n)^{-1}(u), (f_\omega^n)^{-1}(v)\right)\right] \\
&= \sup_{u,v\in S^1} \mathbb{E}\left[d\left(f_{\omega_n}^{-1} \circ \cdots \circ f_{\omega_0}^{-1}(u), f_{\omega_n}^{-1} \circ \cdots \circ f_{\omega_0}^{-1}(v)\right)\right] \leq e^{-\lambda_- n}
\end{aligned}$$

where we have used that the f_{ω_j} are independent and identically distributed in the last equality. Hence the inequality

$$\sup_{x\in S^1} \mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), \sigma(\omega)\right)\right] \leq e^{-\lambda_- n} + \mathbb{E}\left[d\left((f_\omega^k)^{-1}(y), \sigma(\omega)\right)\right] \tag{3.4.9}$$

holds, and by integrating (3.4.9) in $\mathrm{d}\nu(y)$ we conclude that

$$\begin{aligned}
\sup_{x\in S^1} \mathbb{E}\left[d\left((f_\omega^n)^{-1}(x), \sigma(\omega)\right)\right] &\leq e^{-\lambda_- n} + \mathbb{E}\left[\int_{S^1} d\left((f_\omega^k)^{-1}(y), \sigma(\omega)\right) \mathrm{d}\nu(y)\right] \\
&= e^{-\lambda_- n} + \mathbb{E}\left[\int_{S^1} d(y, \sigma(\omega))\, \mathrm{d}(f_\omega^k)^{-1}\nu(y)\right].
\end{aligned}$$

The dominated convergence theorem and Theorem 3.2.3, (ii) imply that

$$\mathbb{E}\left[\int_{S^1} d(y,\sigma(\omega))\,\mathrm{d}(f_\omega^k)^{-1}\nu(y)\right] \xrightarrow[k\to\infty]{} \mathbb{E}\left[\int_{S^1} d(y,\sigma(\omega))\,\mathrm{d}\delta_{\sigma(\omega)}(y)\right] = 0,$$

so (3.4.7) holds. □

3.4.2 Proof of Theorem C

The proof of Theorem E (when the subgroups of $\mathrm{Homeo}_0(S^1)$ act proximally) involves trying to find for a given $n \in \mathbb{N}$ small disjoint open intervals $U, V \subseteq S^1$ containing $\sigma(\omega)$ and $f_\omega^n(0)$ respectively such that $f_\omega^n(S^1 \setminus U) \subseteq V$. Here, the diameters of U, V depend on ω but not on n. In this sense, $\sigma(\omega)$ and $f_\omega^n(0)$ are a repeller-attractor pair for f_ω^n in a weak sense that is sufficient for the proof of the qualitative statement in Theorem E.

On the other hand, to show Theorem C we need to show that $\mathbb{P}\left[f_\omega^n(S^1 \setminus U_n) \subseteq V_n\right]$ is exponentially close to 1 as n increases, where U_n and V_n are disjoint intervals centered around $\sigma(\omega)$ and $f_\omega^n(0)$ respectively such that $\mathrm{diam}(U_n)$, $\mathrm{diam}(V_n)$ are exponentially small in n. The fact that this contraction takes place does not follow from the definition of $\sigma(\omega)$, since $\sigma(\omega)$ is defined by an asymptotic condition saying that $\mathbb{P}$-almost surely any fixed closed interval inside $S^1 \setminus \{\sigma(\omega)\}$ is eventually contracted by f_ω^n. Nevertheless, the following proposition shows that $\sigma(\omega)$ and $f_\omega^n(0)$ are a repeller-attractor pair for f_ω^n in a strong sense suitable for our purposes. This is one of the main points where the strategy deviates from the linear case.

Proposition 3.4.4. *There exists $\varepsilon \in (0,1)$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[f_\omega^n\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \textit{ is not contained in } f_\omega^n(0)^{\varepsilon^n}\right] < 0.$$

Proof. Let $\lambda > 0$, $N \in \mathbb{N}$ be the constants given by Theorem 3.4.2, so

$$\sup_{x,y\in S^1} \mathbb{E}\left[d\left(f_\omega^n(x), f_\omega^n(y)\right)\right] \leq e^{-\lambda n} \tag{3.4.10}$$

for all $n \geq N$.

For $n \geq N$ define $K_n = \lfloor e^{\lambda n/3} \rfloor$, the grid $x_k^n = k/K_n \in S^1$, $0 \leq k \leq K_n - 1$ and the event

$$V_n = \{\omega \in \Omega : d(f_\omega^n(x_k^n), f_\omega^n\left(x_{k+1}^n\right)) \leq e^{-\lambda n/2} \text{ for all } 0 \leq k \leq K_n - 1\},$$

so we have

$$\mathbb{P}\left[V_n^c\right] \leq \sum_{k=1}^{K_n} \mathbb{P}\left[d\left(f_\omega^n(x_k^n), f_\omega^n(x_{k+1}^n)\right) \geq e^{-\lambda n/2}\right] \leq e^{-\lambda n/2} K_n \leq e^{-\lambda n/6},$$

where we have used the Markov inequality and (3.4.10).

Notice that if $\omega \in V_n$, then there exists a unique interval $J_{n,\omega} \subseteq S^1$ of the form $[x_j^n, x_{j+1}^n)$ such that $\mathrm{diam}\left(f_\omega^n(J_{n,\omega})\right) \geq 1 - e^{-\lambda n/2}$.

Claim. *There exists $C > 0$ such that $\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega)) \mid V_n\right] \leq Ce^{-\lambda_1 n}$ for all sufficiently large $n \in \mathbb{N}$, where $\lambda_1 = \max\{\lambda/6, \lambda_-\}$ and λ_- is given by Proposition 3.4.3.*

Proof of the claim. Given $\omega \in V_n$, define $j_{n,\omega} \in S^1$ by

$$j_{n,\omega} = \begin{cases} (f_\omega^n)^{-1}(0) & \text{if } 0 \in f_\omega^n(J_{n,\omega}) \\ (f_\omega^n)^{-1}(1/2) & \text{otherwise.} \end{cases}$$

Since $\operatorname{diam}(f_\omega^n(J_{n,\omega})) \geq 1 - e^{-\lambda n/2}$, for all $n > 2\log 2/\lambda$ we have that $f_\omega^n(J_{n,\omega})$ contains 0 or 1/2, so $j_{n,\omega} \in J_{n,\omega}$. Thus

$$\begin{aligned}\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega)) \mid V_n\right] &\leq \mathbb{E}\left[\operatorname{diam}(J_{n,\omega}) + d(j_{n,\omega}, \sigma(\omega)) \mid V_n\right] \\ &\leq \frac{1}{K_n} + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right) \mid V_n\right] + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(1/2), \sigma(\omega)\right) \mid V_n\right] \\ &\leq \frac{1}{K_n} + \mathbb{P}\left[V_n\right]^{-1}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right)\right] + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(1/2), \sigma(\omega)\right)\right]\right)\end{aligned}$$

for $n > \max\{2\log 2/\lambda, N\}$. From the bound (3.4.7) and the fact that $\mathbb{P}[V_n]$ is bounded away from 0 we obtain the conclusion. □

Let $C, \lambda_1 > 0$ be the constants given by the previous claim and take $\varepsilon \in (e^{-\lambda_1}, 1)$, so that

$$\begin{aligned}\mathbb{P}\left[f_\omega^n\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n}\right] &\leq \mathbb{P}\left[f_\omega^n\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] + \mathbb{P}\left[V_n^c\right] \\ &\leq \mathbb{P}\left[J_{n,\omega} \not\subseteq \sigma(\omega)^{\varepsilon^n} \text{ or } f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] + \mathbb{P}\left[V_n^c\right].\end{aligned} \tag{3.4.11}$$

for all $n \in \mathbb{N}$ large enough. Since $\varepsilon > e^{-\lambda/3}$ and $K_n = \lfloor e^{\lambda n/3} \rfloor$, there exists a constant $C' > 0$ such that the inequalities

$$\begin{aligned}\mathbb{P}\left[J_{n,\omega} \not\subseteq \sigma(\omega)^{\varepsilon^n} \,\middle|\, V_n\right] &\leq \mathbb{P}\left[d(J_{n,\omega}, \sigma(\omega)) \geq \varepsilon^n - \operatorname{diam}(J_{n,\omega}) \,|\, V_n\right] \\ &\leq \frac{\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega) \,|\, V_n\right]}{\varepsilon^n - 1/K_n} \\ &\leq C\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n \left(\frac{1}{1 - 1/(\varepsilon^n K_n)}\right) \leq C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n\end{aligned} \tag{3.4.12}$$

hold, so the right-hand side of (3.4.12) decreases with exponential speed towards 0 by the choice of ε. Here, the first inequality follows from the fact that whenever $\operatorname{diam}(J_{n,\omega}) + d(J_{n,\omega}, \sigma(\omega)) \leq \varepsilon^n$ then necessarily $J_{n,\omega}$ is included in $\sigma(\omega)^{\varepsilon^n}$.

Similarly, since $\varepsilon^n \geq e^{-\lambda n/6} \geq \operatorname{diam}\left(f_\omega^n\left(S^1 \setminus J_{n,\omega}\right)\right)$ for all sufficiently large $n \in \mathbb{N}$, we see that

$$\begin{aligned}\mathbb{P}\left[f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] &\leq \mathbb{P}\left[0 \in J_{n,\omega} \,|\, V_n\right] \\ &\leq \mathbb{P}\left[0 \in J_{n,\omega} \text{ and } J_{n,\omega} \subseteq \sigma(\omega)^{\varepsilon^n} \,\middle|\, V_n\right] + C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n \\ &\leq \mathbb{P}\left[d(0, \sigma(\omega)) \leq \varepsilon^n \,|\, V_n\right] + C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n,\end{aligned}$$

where we have used (3.4.12) in the second inequality. Moreover, Theorem 3.2.6 applied to the distribution of σ (which is $\overline{\mu}$-stationary, where $\overline{\mu}(g) = \mu(g^{-1})$ for $g \in G$) provides $C'', \alpha > 0$ such that

$$\mathbb{P}\left[d(0, \sigma(\omega)) \leq \varepsilon^n\right] \leq C'' \varepsilon^{\alpha n},$$

and from (3.4.11) we conclude that

$$\mathbb{P}\left[f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] \leq \mathbb{P}\left[V_n\right]^{-1} C'' \varepsilon^{\alpha n} + C' \left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n. \tag{3.4.13}$$

The bounds (3.4.13) and (3.4.12) show that the right-hand side of (3.4.11) is exponentially small in n. This finishes the proof of the proposition. □

From now on, the rest of the proof of Theorem C follows the strategy of [Aou11]. Recall that $(\overline{f}_\omega^n)_{n \geq 0}$ denotes the inverse random walk $\overline{f}_\omega^n = f_{\omega_0} \circ f_{\omega_1} \circ \cdots f_{\omega_n}$.

Proposition 3.4.5. *For every $t \in (0,1)$ we have*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[d(f_\omega^n(0), \sigma(\omega)) \leq t^n\right] < 0.$$

Proof. We start with a version of [Aou11, Theorem 4.35] (which in turn is inspired by [Gui90, Lemme 8]), which states that the variables $f_\omega^n(0)$ and $\sigma(\omega)$ become asymptotically independent with exponential speed as $n \to \infty$.

Claim. *There exists a random variable $S(\omega) \in S^1$ independent of σ and constants $C, \lambda > 0$ such that for any Lipschitz function $\psi \colon S^1 \times S^1 \to \mathbb{R}$ we have*

$$\left|\mathbb{E}\left[\psi\left(f_\omega^n(0), \sigma(\omega)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega)\right)\right]\right| \leq C e^{-\lambda n} |\psi|_{\mathrm{Lip}}$$

for sufficiently large $n \in \mathbb{N}$, where

$$|\psi|_{\mathrm{Lip}} = \sup_{\substack{x,y,u,v \in S^1 \\ x \neq y \text{ or } u \neq v}} \frac{|\psi(x,u) - \psi(y,v)|}{d(x,y) + d(u,v)}.$$

Proof of the claim. Let $\lambda_-, \lambda_+ > 0$ be the constants given by Proposition 3.4.3. Consider an independent copy $\omega' = (f_{\omega'_n})_{n \geq 0}$ of the process ω (that is, a coupling of $\mathbb{P}$ with itself). Define $S(\omega')$ as the repeller of the random walk $\left(f_{\omega'_n}^{-1} \circ f_{\omega'_{n-1}}^{-1} \circ \cdots \circ f_{\omega'_0}^{-1}\right)_{n \geq 0}$, so that

$$\sup_{x \in S^1} \mathbb{E}\left[d\left(\overline{f}_{\omega'}^n(x), S(\omega')\right)\right] \leq e^{-\lambda_+ n}$$

holds for all large $n \in \mathbb{N}$.

Decompose

$$\left|\mathbb{E}\left[\psi\left(f_\omega^n(0), \sigma(\omega)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega')\right)\right]\right| \leq \triangle_1 + \triangle_2 + \triangle_3 + \triangle_4$$

where

- $\triangle_1 = \left|\mathbb{E}\left[\psi\left(\sigma(\omega), f_\omega^n(0)\right)\right] - \mathbb{E}\left[\psi\left((f_\omega^n)^{-1}(0), f_\omega^n(0)\right)\right]\right|$,
- $\triangle_2 = \left|\mathbb{E}\left[\psi\left((f_\omega^n)^{-1}(0), f_\omega^n(0)\right)\right] - \mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), f_{\omega_n}\circ\cdots\circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right]\right|$,
- $$\begin{aligned}\triangle_3 &= \left|\mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), f_{\omega_n}\circ\cdots\circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right]\right| \\ &= \left|\mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right]\right|, \text{ and}\end{aligned}$$
- $\triangle_4 = \left|\mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega')\right)\right]\right|$.

Proposition 3.4.3 shows that $\triangle_1 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_- n}$, $\triangle_3 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_- n/2}$ and $\triangle_4 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_+ n/2}$ for all large $n \in \mathbb{N}$. Moreover

$$\begin{aligned}\triangle_2 &\le |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0)\right)\right] + \mathbb{E}\left[d\left(f_\omega^n(0), f_{\omega_n}\circ\cdots\circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right]\right) \\ &= |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0)\right)\right] + \mathbb{E}\left[d\left(\overline{f}_\omega^n(0), \overline{f}_\omega^{\lfloor n/2\rfloor}(0)\right)\right]\right) \\ &\le |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right)\right] + \mathbb{E}\left[d\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), \sigma(\omega)\right)\right]\right. \\ &\qquad\left. + \mathbb{E}\left[d\left(\overline{f}_\omega^n(0), T(\omega)\right)\right] + \mathbb{E}\left[d\left(\overline{f}_\omega^{\lfloor n/2\rfloor}(0), T(\omega)\right)\right]\right)\end{aligned}$$

which is at most

$$|\psi|_{\mathrm{Lip}}\left(e^{-\lambda_- n} + e^{-\lambda_- n/2} + e^{-\lambda_+ n} + e^{-\lambda_+ n/2}\right)$$

by Proposition 3.4.3 again. The claim follows. □

For any $\varepsilon \in (0, 1/2)$, take a $1/\varepsilon$-Lipschitz function $\phi_\varepsilon \colon [0,1] \to [0,1]$ such that $\phi\big|_{[0,\varepsilon]} = 1$ and $\phi\big|_{[2\varepsilon,1]} = 0$, so

$$1_{[0,\varepsilon]} \le \phi_\varepsilon \le 1_{[0,2\varepsilon]}$$

holds and $\psi_\varepsilon \doteq \phi_\varepsilon \circ d \colon S^1 \times S^1 \to [0,1]$ is also $1/\varepsilon$-Lipschitz. Let $C, \lambda > 0$ be the constants given by the previous claim.

Now for all $n \in \mathbb{N}$ large enough we have

$$\begin{aligned}\mathbb{P}\left[d\left(f_\omega^n(0), \sigma(\omega)\right) \le t^n\right] \le \mathbb{E}\left[\psi_{t^n}\left(f_\omega^n(0), \sigma(\omega)\right)\right] &\le \mathbb{E}\left[\psi_{t^n}\left(\sigma(\omega), S(\omega)\right)\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}} \\ &\le \mathbb{P}\left[d\left(\sigma(\omega), S(\omega)\right) \le 2t^n\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}} \\ &\le \sup_{x\in S^1} \mathbb{P}\left[d(\sigma(\omega), x) \le 2t^n\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}},\end{aligned}$$

where we have used the independence of σ and S in the last inequality. The first term

$$\sup_{x\in S^1} \mathbb{P}\left[d(\sigma(\omega), x) \le 2t^n\right]$$

is exponentially small in n by Theorem 3.2.6, and the second term

$$Ce^{-\lambda n/2}|\psi_{t^n}|_{\mathrm{Lip}} = C\left(\frac{e^{-\lambda/2}}{t}\right)^n$$

is also exponentially small in n whenever $t > e^{-\lambda/2}$. The proposition is thus proven in this case, and is also true for $t \leq e^{-\lambda/2}$ as a consequence. $\square$

Remark. In the proof of the previous theorem we have abused notation by writing $S(\omega)$: the random variable S is not a function of ω, and is defined on a larger probability space. We have done so as to not to weigh down the notation with distinctions between this larger probability space and its quotient $(\Omega, \mathbb{P})$, since all relevant means and measures of sets coincide with those of the measure $\mathbb{P}$.

Proof of Theorem C. Fix two non-degenerate probability measures μ_1, μ_2 on countable subgroups G_1, G_2 of $\mathrm{Diff}^1_0(S^1)$ acting proximally on S^1 such that μ_1, μ_2 satisfy the moment condition (M).

Claim. *For every $n \in \mathbb{N}$ let $\omega' \in \Omega_2 \mapsto l_n(\omega') \in S^1$ be a measurable map. For every $t \in (0,1)$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \leq t^n\right] < 0 \tag{3.4.14}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(\sigma(\omega), l_n(\omega')) \leq t^n\right] < 0. \tag{3.4.15}$$

Proof of the claim. By independence, we have

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \leq t^n\right] \leq \sup_{x\in S^1} \mathbb{P}_1 \left[d(f^n_\omega(0), x) \leq t^n\right] = \sup_{x\in S^1} \mathbb{P}_1 \left[d(\overline{f}^n_\omega(0), x) \leq t^n\right].$$

Proposition 3.4.3 and the Markov inequality imply that

$$\mathbb{P}_1 \left[d\left(\overline{f}^n_\omega(0), T(\omega)\right) \geq e^{-\lambda_+ n/2}\right] \leq e^{-\lambda_+ n/2},$$

for some $\lambda_+ > 0$ and all large $n \in \mathbb{N}$, and thus

$$\sup_{x\in S^1} \mathbb{P}_1 \left[d\left(\overline{f}^n_\omega(0), x\right) \leq t^n\right] \leq \sup_{x\in S^1} \mathbb{P}_1 \left[d\left(T(\omega), x\right) \leq t^n + e^{-\lambda_+ n/2}\right] + e^{-\lambda_+ n/2}.$$

Take $C, \alpha > 0$ such that the distribution of T is (C,α)-Hölder, so

$$\sup_{x\in S^1} \mathbb{P}_1 \left[d(T(\omega), x) \leq t^n + e^{-\lambda_+ n/2}\right] \leq C(t^n + e^{-\lambda_+ n/2})^\alpha$$

and

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \leq t^n\right] \leq C(t^n + e^{-\lambda_+ n/2})^\alpha + e^{-\lambda_+ n/2}$$

is exponentially small in n. This gives (3.4.14), and (3.4.15) follows in the same way. $\square$

Take $\varepsilon \in (0,1)$ so that the conclusion of Proposition 3.4.4 is verified for $\mathbb{P}_1 = \mu_1^{\otimes\mathbb{N}}$ and $\mathbb{P}_2 = \mu_2^{\otimes\mathbb{N}}$. Given $\omega \in \Omega_1$, $\omega' \in \Omega_2$ and $n \in \mathbb{N}$ we say that f^n_ω *and* $f^n_{\omega'}$ *are in ε-transverse position at time n* if the intervals $f^n_\omega(0)^{\varepsilon^n}$, $f^n_{\omega'}(0)^{\varepsilon^n}$, $\sigma(\omega)^{\varepsilon^n}$ and $\sigma(\omega')^{\varepsilon^n}$ are pairwise disjoint. This is exactly the situation in Figure 3.1 for

$$I_{n,\omega} = f^n_\omega(0)^{\varepsilon^n}, \quad I_{n,\omega'} = f^n_{\omega'}(0)^{\varepsilon^n}, \quad J_{n,\omega} = \sigma(\omega)^{\varepsilon^n} \text{ and } J_{n,\omega'} = \sigma(\omega')^{\varepsilon^n}.$$

Proposition 3.4.5 shows that the probability that the pair of sets $(f^n_\omega(0)^{\varepsilon^n}, \sigma(\omega)^{\varepsilon^n})$ intersect or that the pair $(f^n_{\omega'}(0)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n})$ intersect is exponentially small in n. The previous claim shows that the probability that the remaining pairs

$$(f^n_\omega(0)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n}), \quad (f^n_{\omega'}(0)^{\varepsilon^n}, \sigma(\omega)^{\varepsilon^n}), \quad (f^n_\omega(0)^{\varepsilon^n}, f^n_{\omega'}(0)^{\varepsilon^n}) \text{ or } (\sigma(\omega)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n})$$

intersect is exponentially small in n. We conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[f^n_\omega \text{ and } f^n_{\omega'} \text{ are not in } \varepsilon\text{-transverse position} \right] < 0. \qquad \square$$

Chapter 4

The Poisson boundary of Thompson's group T is not the circle

This chapter corresponds to the preprint [GVKS25], **and represents joint work with Cosmas Kravaris (Princeton University) and Eduardo Silva (Universität Münster)**.

Let μ be a non-degenerate probability measure with finite entropy on a countable group G of orientation-preserving homeomorphisms of the circle S^1 acting proximally, minimally and topologically non-freely on S^1. We prove that the circle S^1 endowed with its unique μ-stationary probability measure is not the Poisson boundary of (G, μ). An alternative proof is given for groups of piecewise affine homeomorphisms of S^1. When G is Thompson's group T and μ is finitely supported, this answers a question asked by B. Deroin [Der13b] and A. Navas [Nav18].

4.1 Introduction

Let G be a countable group and μ a probability measure on G that we will systematically assume to be *non-degenerate*, that is, with support generating all of G as a semigroup. First defined by H. Furstenberg [Fur63], the *Poisson boundary* of (G, μ) is a probability space $(\partial_\mu G, \nu_\mu)$ that captures all the stochastically significant asymptotic behavior of sample paths of the μ-random walk on G. The pair $(\partial_\mu G, \nu_\mu)$ is equipped with a non-singular action of G such that ν_μ is *μ-stationary*, that is, such that the equality $\nu_\mu = \sum_{g\in G} \mu(g) g_* \nu_\mu$ holds, and can be characterized as follows. A *μ-boundary* of G is a probability space (X, ν) equipped with a measurable G-action and a shift-invariant, G-equivariant *boundary map* $\xi\colon G^{\mathbb{N}} \to X$ on the space of trajectories $(G^{\mathbb{N}}, \mathbb{P})$ of the right μ-random walk with $\xi_* \mathbb{P} = \nu$. The Poisson boundary $(\partial_\mu G, \nu_\mu)$ is the *maximal* μ-boundary, in the sense that any μ-boundary is a G-equivariant measurable quotient of it, and is unique up to G-equivariant isomorphisms.

In this chapter we are interested in random walks on countable subgroups of the group $\mathrm{Homeo}_0(S^1)$ of orientation-preserving homeomorphisms of the circle S^1. An action of a countable group G on S^1 by homeomorphisms is called

- *minimal* if all the G-orbits are dense in S^1,
- *proximal* if for every proper closed interval $I \subsetneq S^1$ and all $\varepsilon > 0$ there is $g \in G$ with diameter $\mathrm{diam}(g(I)) < \varepsilon$, and
- *non-elementary* if there is no G-invariant probability measure on S^1.

A theorem of G. Margulis [Mar00] (see also [Ant84]) states that for any non-elementary subgroup G of $\mathrm{Homeo}_0(S^1)$ there exists a G-equivariant quotient of S^1, still homeomorphic to S^1, on which G acts minimally and proximally. Thus, the study of any non-elementary group action on S^1 often reduces to the study of a minimal and proximal action (see Subsection 4.2.4 for a precise statement).

Let G be a countable group acting proximally and minimally on S^1, and let μ be a non-degenerate probability measure on G. Deroin-Kleptsyn-Navas [DKN07] show that there exists a unique μ-stationary probability measure ν on S^1, and that for $\mathbb{P}$-almost every sample path $\mathbf{w} = (w_n)_{n\geq 0} \in G^{\mathbb{N}}$ of the right μ-random walk on G there exists a point $\xi(\mathbf{w}) \in S^1$ such that $\lim_{n\to\infty}(w_n)_*\nu = \delta_{\xi(\mathbf{w})}$ in the weak-$*$ topology, where $\delta_{\xi(\mathbf{w})}$ denotes the Dirac mass at $\xi(\mathbf{w})$. The measure ν coincides with the distribution of $\xi(\mathbf{w})$ on S^1, and therefore the space (S^1, ν) provides a μ-boundary of G.

The action of a cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$ on the boundary $\partial\mathbb{H}^2 \cong S^1$ of the hyperbolic plane is proximal and minimal, and for any non-degenerate finitely supported μ the pair (S^1, ν) is a topological model for the Poisson boundary since it coincides with the hitting measure on their Gromov boundaries (see the references in Subsection 4.1.1). A natural question is to determine the largest class of subgroups for which this holds, and a particular case, posed by B. Deroin [Der13b, Section 6] and by A. Navas [Nav18, Question 19], is whether this holds for Thompson's group T of piecewise dyadically affine homeomorphisms of $S^1 \cong \mathbb{R}/\mathbb{Z}$. Here, an orientation-preserving homeomorphism $g\colon S^1 \to S^1$ belongs to T if and only if there exists a finite subset of dyadic rationals $D \subseteq \mathbb{Z}[1/2]/\mathbb{Z}$ such that g restricted to every connected component C of $S^1 \setminus D$ is of the form $g(x) = 2^k x + b$, $x \in S^1$, for some $k \in \mathbb{Z}$ and $b \in \mathbb{Z}[1/2]/\mathbb{Z}$. This group was introduced by R. Thompson in unpublished notes [Tho65] as the first example of a finitely presented infinite simple group, and has been extensively studied from algebraic, dynamical, and cohomological viewpoints, see, e.g., [Bri96, GS87, BG84].

We answer this question in the negative by finding a general obstruction for this to be the case. We say that an action of a subgroup G of $\mathrm{Homeo}_0(S^1)$ on S^1 is *topologically non-free* if there exists $g \in G \setminus \{e_G\}$ whose set of fixed points has non-empty interior. Notice that the action of Thompson's group T on S^1 is topologically non-free.

Theorem F. *Let G be a countable subgroup of $\mathrm{Homeo}_0(S^1)$ whose action on S^1 is minimal, proximal and topologically non-free, and let μ be a finite-entropy non-degenerate probability measure on G. Then (S^1, ν) is not the Poisson boundary of (G, μ).*

Theorem F is related to the well-known open problem [CFP96] on whether Thompson's group F, the group of piecewise dyadically affine homeomorphisms of the interval $[0,1]$, is amenable. Indeed, the action of a countable group G on its Poisson boundary $(\partial_\mu G, \nu_\mu)$ is amenable [Zim78], and hence for ν_μ-almost every $x \in \partial_\mu G$ the stabilizer subgroup $G_x \subseteq G$ is amenable (see [ADR00, Corollary 5.3.33]). If the circle were the Poisson boundary of T then we would conclude that F is amenable, since for each $x \in S^1$ the stabilizer $T_x \subseteq T$ contains a copy of F. Theorem F implies that this strategy does not work whenever the probability measure μ has finite entropy.

The proof of Theorem F is sketched in Subsection 4.1.1, and relies on the conditional entropy criterion of V. Kaimanovich [Kai85] together with a conditional version of a method used by A. Erschler to show positivity of asymptotic entropy [Ers04]. As is often the case with entropy methods for Poisson boundaries, our proof does not provide an explicit μ-boundary that is not a G-equivariant quotient of (S^1, ν) (or, equivalently, explicit bounded μ-harmonic functions that do not arise from the μ-boundary (S^1, ν), see Subsection 4.2.1). Our second theorem gives precisely this information for a more restricted class of subgroups of $\mathrm{Homeo}_0(S^1)$, which still includes Thompson's group T.

We denote by $\mathrm{PAff}_0(S^1)$ the group of piecewise affine orientation-preserving homeomorphisms of $S^1 \cong \mathbb{R}/\mathbb{Z}$ whose derivative has finitely many discontinuity points. Given $g \in \mathrm{PAff}_0(S^1)$ denote by $\mathbf{Br}_g \subseteq S^1$ the (finite) set of discontinuities of its derivative, which we call the *breakpoints* of g. For a countable subgroup G of $\mathrm{PAff}_0(S^1)$ with a minimal, proximal and topologically non-free action on S^1, we define $\mathbf{Br} = \bigcup_{g \in G} \mathbf{Br}_g$ the *set of breakpoints of* G and show that, with respect to an appropriate action of G, there is a μ-stationary probability measure $\widetilde{\nu}$ on $\mathbb{R}^{\mathbf{Br}}$ such that $(\mathbb{R}^{\mathbf{Br}}, \widetilde{\nu})$ is a μ-boundary of G (see Subsection 4.1.2). We call this space the *breakpoint boundary* of G. This construction is analogous to the boundary of Thompson's group F constructed by V. Kaimanovich in [Kai17].

Theorem G. *Let G be a countable subgroup of $\mathrm{PAff}_0(S^1)$ whose action on S^1 is minimal, proximal, and topologically non-free, and let μ be a non-degenerate probability measure on G such that $\sum_{g \in G} \mu(g)|\mathbf{Br}_g| < \infty$. Then the breakpoint boundary $(\mathbb{R}^{\mathbf{Br}}, \widetilde{\nu})$ is not a G-equivariant quotient of (S^1, ν). In particular, (S^1, ν) is not the Poisson boundary of (G, μ).*

The class of groups to which Theorem F applies is larger than that considered in Theorem G. Indeed, note that $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$ acts minimally and proximally on S^1 through its natural projective action, and it follows from [Cor21, Theorem 1.4] that it does not embed into $\mathrm{PAff}_0(S^1)$. Consider the group G of all piecewise-$\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$ homeomorphisms of S^1 with breakpoints in the set of fixed points of hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$. Then G is countable, satisfies the hypotheses of Theorem F and is not conjugate to a subgroup of $\mathrm{PAff}_0(S^1)$ since it contains $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$.

4.1.1 Further background on identification of Poisson boundaries

Given a probability measure μ on a countable group G, a natural question is to find an explicit model for the Poisson boundary of (G, μ). That is, to identify the Poisson boundary of the random walk with a concrete μ-boundary expressed in terms of geometric, combinatorial, or algebraic properties of G. Currently, the main tools for studying this question are based on entropy. For a probability measure μ on a group G, the *Shannon entropy* of μ is defined as $H(\mu) = -\sum_{g\in G}\mu(g)\log\mu(g)$. The *asymptotic entropy* of a probability measure μ is defined by $h(\mu) = \lim_{n\to\infty} H(\mu^{*n})/n$. This quantity was introduced by A. Avez [Ave72], who proved that if μ is finitely supported and $h(\mu) = 0$, then the Poisson boundary of (G,μ) is trivial. The *entropy criterion* of Y. Derriennic [Der80] and Kaimanovich-Vershik [KV83] states that if $H(\mu) < \infty$, then $h(\mu) > 0$ if and only (G,μ) has a non-trivial Poisson boundary. This result was later extended by V. Kaimanovich, who proved that a μ-boundary of G is the Poisson boundary if and only if the sequence of *mean conditional* entropies at time n of μ (see Definition 4.2.4) grows sublinearly [Kai85, Kai00] (see also Theorem 4.2.5 below). It is important to note that these criteria have two main restrictions: the first is that the hypothesis $H(\mu) < \infty$ is crucial and the criteria do not work in a general context for measures with infinite entropy. The second restriction is that one needs to identify by other means a μ-boundary of G that serves as a potential candidate for the Poisson boundary. There are families of groups for which one can prove that some random walks on them have a non-trivial Poisson boundary by using the entropy criterion but for which there is no known non-trivial μ-boundary. An example of this is the wreath product $\mathbb{Z}/2\mathbb{Z}\wr\mathbb{Z}^3$, which has non-trivial Poisson boundary for each non-degenerate probability measure with finite entropy [Ers04].

Next, we recall results on the identification of the Poisson boundary for two families of groups: Gromov-hyperbolic groups and wreath products. For a more comprehensive list of results regarding the identification of Poisson boundaries of random walks on groups, we refer to [Ers10] and [SM24, Section 3.3.5].

Gromov-hyperbolic groups

The first complete description of non-trivial Poisson boundaries goes back to Dynkin-Maljutov [DM61], who proved that for any non-abelian free group and any probability measure supported on a free generating set, the corresponding Poisson boundary can be identified with the space of infinite reduced words of the free group endowed with its unique stationary measure. This result was extended to all probability measures with finite support by Y. Derriennic [Der75]. More generally, A. Ancona [Anc87] proved that the Poisson boundary of a non-elementary Gromov-hyperbolic group with respect to a finitely supported measure coincides with its Gromov boundary endowed with the unique stationary measure. V. Kaimanovich later used the conditional entropy criterion to generalize the latter result to hold for any non-degenerate probability measure with finite entropy and finite first logarithmic moment [Kai00]. Recently, this description was proved to hold for all non-degenerate measures with finite entropy by Chawla-Forghani-Frisch-Tiozzo [CFFT25].

Wreath products

Given countable groups A and B, the *wreath product* $A\wr B$ is the semidirect product $(\bigoplus_B A)\rtimes B$ where B acts on $\bigoplus_B A$ by translations. The main tool used to exhibit a non-trivial μ-boundary of random walks on wreath products is the *stabilization of lamp configurations*: suppose that μ is a probability measure on $A \wr B$ such that for $\mathbb{P}$-almost every trajectory $\{(\varphi_n, X_n)\}_{n\geq 0}$ of the μ-random walk on $A \wr B$ and for each $b \in B$, there exists $N \geq 1$ such that for every $n \geq N$, we have $\varphi_n(b) = \varphi_N(b)$. From this one can deduce the existence of a μ-stationary probability measure ν on $\prod_B A$ such that $(\prod_B A, \nu)$ is a μ-boundary of $A \wr B$. A sufficient condition that guarantees the stabilization of lamp configurations is that μ has a finite first moment with respect to a word metric on $A \wr B$ and that the projection of μ to B defines a transient random walk (see [Ers11, Lemma 1.1], following [KV83, Kai01, KW07]). In particular for $B = \mathbb{Z}^d$, $d \geq 1$, if $H(\mu) < \infty$ and μ satisfies the stabilization of lamp configurations, the space $(\prod_B A, \nu)$ is the Poisson boundary of $(A \wr B, \mu)$ (see [FS23], following [Kai01, Ers11, LP21]). In contrast, there are non-degenerate probability measures on $A \wr \mathbb{Z}^d$, $d \geq 3$, with an infinite first moment and finite entropy such that the lamp configuration does not stabilize, and yet the Poisson boundary is non-trivial [Kai83, Ers11, LP21]. For such probability measures, there are no known constructions of non-trivial μ-boundaries.

A natural boundary for Thompson's group T

The μ-boundary (S^1, ν) can be considered as a "natural" candidate for the Poisson boundary of Thompson's group T in the following sense. A faithful action $G \curvearrowright X$ of a group G on a locally compact Hausdorff perfect space X by homeomorphisms is a *Rubin action* if for every open subset $U \subseteq X$ and every $x \in U$, the closure of the orbit of x under the action of the subgroup

$$\{g \in G : g\big|_{X\setminus U} = \mathrm{id}_{X\setminus U}\}$$

contains a neighborhood of x. A theorem by M. Rubin [Rub89] states that there exists a unique Rubin action up to G-equivariant homeomorphisms (see also [BEM25] for a recent short proof). One can phrase this result as saying that a group admits at most one "sufficiently rich" *micro-supported* action on a compact space. Notice that the action of Thompson's group T on S^1 is Rubin.

To the best of our knowledge, Thompson's group T is the first example of a group such that the Poisson boundary of a simple random walk is strictly larger than a "natural" candidate μ-boundary. One can compare this result with the recent work of Chawla-Frisch [CF25], from which it follows that on any non-abelian free group there are probability measures with infinite entropy such that the Poisson boundary is not the Gromov boundary of the free group.

4.1.2 Structure of the proofs

The proof of Theorem F follows similar steps to a method used by A. Erschler in [Ers04] (see also Theorem 4.3.1) to show that $h(\mu) > 0$ for a probability measure μ on a group G. This method consists of verifying that the sequence $(H(\mu^{*n}))_{n\geq 0}$ grows linearly under the following condition: there exist $p, c \in (0, 1)$ and $a \in \mathrm{supp}(\mu) \setminus \{e_G\}$ such that for every $n \in \mathbb{N}$, with probability at least p we can choose elements $b_1, \dots, b_{k+1} \in G$ and times $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ with

$k \geq cn$ such that the random walk w_n at step n can be expressed *in a unique way* as

$$w_n = b_1\varepsilon_1 b_2 \cdots b_k \varepsilon_k b_{k+1},$$

where $\varepsilon_j \in \{a, e_G\}$ is the increment of the random walk at time i_j for every $1 \leq j \leq k$.

We state a conditional version of Erschler's method in Theorem 4.3.2, which gives sufficient conditions to ensure that the sequence of conditional entropies at time n with respect to a given μ-boundary $\mathbf{X}$ grows linearly on n. By Kaimanovich's conditional entropy criterion and the hypothesis of $H(\mu) < \infty$ we conclude that the space $\mathbf{X}$ not the Poisson boundary of (G, μ). The proof of Theorem F consists on applying Theorem 4.3.2 to the μ-boundary (S^1, ν), and it follows the next steps.

- The main dynamical input that guarantees that we can apply Theorem 4.3.2 in the context of Theorem F is Proposition 4.6.2, which asserts that for each sufficiently small interval $I \subseteq S^1$, there are linearly many intervals in the sequence $I, w_1(I), \ldots, w_n(I)$ such that each of these intervals *dominates* all the ones preceding it. Here, when $I_1, I_2 \subseteq S^1$ are closed intervals we say that I_1 *dominates* I_2 if they are either disjoint or if the interior of I_1 contains I_2.

- By the hypotheses on the action of G on S^1, Proposition 4.2.10 ensures that there exists a non-trivial element $a \in G \setminus \{e_G\}$ such that $\overline{\{x \in S^1 : a(x) \neq x\}} \subseteq I$.

- The non-degeneracy assumption on μ guarantees that, up to replacing μ with a convolution power μ^{*s} for some $s \geq 1$, we may assume that a belongs to the support of μ. From the above, we obtain $c \in (0,1)$ such that, in expectation, there are at least $k \geq cn$ times $1 \leq i_1 < \cdots < i_k \leq n$ such that $w_{i_j-1}(I)$ dominates $w_{i_l-1}(I)$ and $w_{i_j-1}^{-1} w_{i_j} \in \{a, e_G\}$ for all $1 \leq l < j \leq k$.

- Given $n \in \mathbb{N}_+$, fix a trajectory $\mathbf{w}$ such that there are at least $k \geq cn$ times $1 \leq i_1 < \cdots < i_k \leq n$ as in the previous step. Write $w_n = b_1\varepsilon_1 b_2 \cdots b_k\varepsilon_k b_{k+1}$ where the $\varepsilon_1, \ldots, \varepsilon_k \in \{a, e_G\}$ are the jumps at times $i_1, \ldots, i_k$. Then, Lemma 4.4.2 says that whenever $\varepsilon'_1, \ldots, \varepsilon'_k$ range over $\{a, e_G\}$, the resulting elements $b_1\varepsilon'_1 b_2 \cdots b_k\varepsilon'_k b_{k+1}$ are pairwise distinct. From this, we verify the hypotheses of Theorem 4.3.2 and finish the proof.

The proof of Theorem G in Section 4.8 is different and does not rely on entropy techniques. Given a countable subgroup G of $\mathrm{PAff}_0(S^1)$, the set $\mathbf{Br}$ of breakpoints of elements of G is countable, and there is a map $\mathcal{C}\colon G \to \mathbb{R}^{\mathbf{Br}}$ given by

$$\mathcal{C}_g(x) = \log\big((g^{-1})'(x^+)\big) - \log\big((g^{-1})'(x^-)\big) \text{ for each } x \in \mathbf{Br},$$

that records the discontinuities of the derivative of $g \in G$. The map $\mathcal{C}\colon G \to \mathbb{R}^{\mathbf{Br}}$ is an additive cocycle for the natural action of G on $\mathbb{R}^{\mathbf{Br}}$ obtained from the action of G on $\mathbf{Br}$ (see Equation (4.8.1)). The breakpoint boundary is defined as follows.

- Lemma 4.8.2 shows that the trajectory in $\mathbf{Br}$ of any breakpoint through the μ-random walk is transient. This relies on a general comparison lemma for Markov operators [BLP77].

- For $\mathbb{P}$-almost every $\mathbf{w} = (w_n)_{n\geq 0}$ the configurations $\mathcal{C}_{w_n}$ converge pointwise to a configuration $\mathcal{C}_\infty(\mathbf{w}) \in \mathbb{R}^{\mathbf{Br}}$ as $n \to \infty$. This defines an associated hitting measure $\widetilde{\nu}$ on $\mathbb{R}^{\mathbf{Br}}$,

- We define another action of G on $\mathbb{R}^{\mathbf{Br}}$ by $(g.\mathcal{C})(x) = \mathcal{C}_g(x) + \mathcal{C}(g^{-1}x)$ for every $x \in \mathbf{Br}$, $\mathcal{C} \in \mathbb{R}^{\mathbf{Br}}$ and $g \in G$. With respect to this action, the space $(\mathbb{R}^{\mathbf{Br}}, \widetilde{\nu})$ is a μ-boundary of G.

Using the breakpoint boundary $(\mathbb{R}^{\mathbf{Br}}, \widetilde{\nu})$, we find for each $n \geq 1$ a bounded μ-harmonic function $\zeta_n \colon G \to \mathbb{R}$ and an element $a_n \in G$ with small support such that there is a constant $K > 0$ with $|\zeta_n(a_n) - \zeta_n(e_G)| > K$ for all $n \in \mathbb{N}$. This cannot occur for bounded harmonic functions obtained from (S^1, ν).

We remark that the construction outlined in the first two steps of the proof was used by B. Stankov to prove that, under the same moment condition on μ as in Theorem G, any subgroup of Monod's non-amenable group $H(\mathbb{Z})$ of piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ homeomorphisms of the line is either locally solvable or has non-trivial Poisson boundary [Sta21]. Also, the original construction of the breakpoint boundary in [Kai17] draws from the explicit geometry of the Schreier graphs of the action of F on the interval, which were completely described by D. Savchuk [Sav15].

4.1.3 Organization of the chapter

In Section 4.2 we recall background material on random walks on groups, Poisson boundaries, the conditional entropy criterion, and properties of groups acting on the circle. In Section 4.3 we describe the method of A. Erschler (Theorem 4.3.1) used to prove positivity of asymptotic entropy and our conditional version of it (Theorem 4.3.2). In Section 4.4 we prove Lemma 4.4.2 and Proposition 4.4.5, which give sufficient conditions for us to verify the hypotheses of Theorem 4.3.2. Next, in Section 4.5 we prove two quantitative statements for random walks on S^1 that are key to our results: Corollary 4.5.2 and Proposition 4.5.3. Afterwards, in Section 4.6 we apply them to prove Proposition 4.6.2 that guarantees that in expectation there will be linearly many dominating intervals, and use it in Section 4.7 to prove Theorem F. Finally, in Section 4.8 we define the breakpoint boundary and prove Theorem G.

4.1.4 Acknowledgements

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4.2 Preliminaries

We review some relevant background on Poisson boundaries of random walks and groups acting on the circle. In this section G is a countable group and μ is a non-degenerate probability measure on G.

4.2.1 Random walks on groups and Poisson boundaries

Consider the probability measure $\mathbb{P}$ obtained as the push-forward of the Bernoulli measure $\mu^{\otimes\mathbb{N}}$ through the map

$$\begin{aligned} G^{\mathbb{N}} &\to G^{\mathbb{N}} \\ (g_1, g_2, g_3, \ldots) &\mapsto (w_0, w_1, w_2, w_3, \ldots) = (e_G, g_1, g_1g_2, g_1g_2g_3, \ldots). \end{aligned}$$

The space $(G^{\mathbb{N}}, \mathbb{P})$ is called the *space of sample paths* or the *space of trajectories* of the right μ-random walk. We denote by $\sigma\colon G^{\mathbb{N}} \to G^{\mathbb{N}}$ the *shift map* $\sigma((w_n)_{n\geq 0}) = (w_{n+1})_{n\geq 0}$.

The Poisson boundary of the μ-random walk on G was already defined in the introduction as the maximal μ-boundary of G. An alternative definition, due to Zimmer [Zim78], is the following.

Definition 4.2.1. *Two sample paths $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{N}}$ are said to be equivalent if there exist $i, j \geq 0$ such that $\sigma^i(\mathbf{w}) = \sigma^j(\mathbf{w}')$. Consider the measurable hull associated with this equivalence relation, that is, the σ-algebra formed by all measurable subsets of the space of trajectories $(G^{\mathbb{N}}, \mathbb{P})$ which are unions of the equivalence classes of $\sim$ up to $\mathbb{P}$-null sets. The associated quotient space is called the* Poisson boundary *of the random walk (G, μ).*

For further equivalent definitions of the Poisson boundary we refer to the articles [KV83], [Kai00, Section 1] and the references therein. For an overview of the study of Poisson boundaries and random walks on groups we refer to the surveys [Fur02, Ers10, Zhe23].

Denote by $H^\infty(G, \mu)$ the space of *bounded harmonic functions*, that is, of bounded functions $\zeta\colon G \to \mathbb{R}$ such that $\zeta(g) = \sum_{h\in G} \zeta(gh)\mu(h)$ for all $g \in G$. The vector space $H^\infty(G, \mu)$ becomes a Banach space by equipping it with the ℓ^∞-norm. For a probability space (X, ν) endowed with a measurable action of G, we say that ν is *μ-stationary* if $\nu = \mu * \nu = \sum_{g\in G} \mu(g) g_*\nu$. In this case, for each $F \in L^\infty(X, \nu)$ one can define a μ-harmonic function $\zeta \in H^\infty(G, \mu)$ by

$$\zeta(g) = \int_X F(x)\,\mathrm{d}g_*\nu = \int_X F(g(x))\,\mathrm{d}\nu, \text{ for each } g \in G.$$

The map $F \in L^\infty(X, \nu) \mapsto \zeta \in H^\infty(G, \mu)$ is called the *Poisson transform associated with* (X, ν).

We recall a standard result on the Poisson transform associated to a μ-boundary, which will be used in Section 4.8 within the proof of Theorem G.

Theorem 4.2.2 ([Fur63], [Aze70, Chapitre I], see [Gla76, Theorem 4.4])**.** *Let (X, ν) be a μ-boundary of G. Then the Poisson transform associated with (X, ν) is an isometry between $L^\infty(X, \nu)$ and a closed subspace of $H^\infty(G, \mu)$, and it is surjective if and only if (X, ν) coincides with the Poisson boundary $(\partial_\mu G, \nu)$.*

4.2.2 Conditional probabilities with respect to a μ-boundary

Let $\mathbf{X} = (X, \nu)$ be a μ-boundary of G. V. Rokhlin's theory of measurable partitions of Lebesgue spaces [Roh67] allows one to disintegrate the probability measure $\mathbb{P}$ with respect to the boundary map $(G^{\mathbb{N}}, \mathbb{P}) \to (X, \nu)$. That is, for ν-almost every $\xi \in X$ there is a probability measure $\mathbb{P}^\xi$

supported on the fiber of ξ in $G^{\mathbb{N}}$ such that $\mathbb{P} = \int_X \mathbb{P}^\xi \, \mathrm{d}\nu(\xi)$, see [Kai00, Section 3]. These conditional probability measures determine Markov chains on G with transition probabilities

$$\mathbb{P}^\xi \left[w_{n+1} = g \mid w_n = h\right] = \mathbb{P}\left[w_{n+1} = g \mid w_n = h\right] \frac{\mathrm{d}g_*\nu}{\mathrm{d}h_*\nu}(\xi) = \mu(h^{-1}g)\frac{\mathrm{d}g_*\nu}{\mathrm{d}h_*\nu}(\xi) \tag{4.2.1}$$

for every $g, h \in G$ and for ν-almost every $\xi \in X$; see [KV83, Section 0.3].

Lemma 4.2.3. *Consider a μ-boundary (X, ν) of G. Let $J \subseteq X$ be a measurable subset and let $a \in G$ be such that for every $x \in X \backslash J$ we have $a(x) = x$. Then, for every $g \in G$ and ν-almost every $\xi \in X \backslash g(J)$ we have $\frac{\mathrm{d}(ga)_*\nu}{\mathrm{d}g_*\nu}(\xi) = 1$. In particular, we have $\mathbb{P}^\xi[w_{n+1} = ga \mid w_n = g] = \mu(a)$ for every $n \geq 1$.*

Proof. Consider an arbitrary measurable subset $A \subseteq X \backslash g(J)$. Then $g^{-1}A \subseteq X \backslash J$, and hence $(ga)^{-1}A = a^{-1}g^{-1}A = g^{-1}A$. From this, we have

$$g_*\nu(A) = \nu(g^{-1}A) = \mathbb{P}\left[\xi(\mathbf{w}) \in g^{-1}A\right] = \mathbb{P}\left[\xi(\mathbf{w}) \in (ga)^{-1}A\right] = (ga)_*\nu(A).$$

Since A was arbitrary, the above implies the first statement of the lemma. The second statement follows from Equation (4.2.1). □

4.2.3 Entropy

We consider countable partitions of the space of sample paths $G^{\mathbb{N}}$ in the definition of entropy below. For every $k \geq 1$, define the partition α_k of the space of sample paths $G^{\mathbb{N}}$ where two trajectories $\mathbf{w} = (w_n)_{n\geq 0}$, $\mathbf{w}' = (w'_n)_{n\geq 0} \in G^{\mathbb{N}}$ belong to the same element of α_k if and only if $w_k = w'_k$. In other words, α_k is the partition defined by fixing the element visited by the random walk at time k.

The *Shannon entropy* of a countable partition ρ of the space of sample paths $G^{\mathbb{N}}$ with respect to the probability measure $\mathbb{P}$ is defined as $H(\rho) = -\sum_{k\geq 1} \mathbb{P}[\rho_k] \log \mathbb{P}[\rho_k]$. Notice that $H(\alpha_1) = H(\mu)$ is the Shannon entropy of the probability measure μ.

Definition 4.2.4. *Let $\rho = \{\rho_k\}_{k\geq 1}$ be a countable partition of the space of sample paths $G^{\mathbb{N}}$ and let $\mathbf{X} = (X, \nu)$ be a μ-boundary of G. Consider the disintegration $\mathbb{P} = \int_X \mathbb{P}^\xi \, \mathrm{d}\nu(\xi)$ with respect to the boundary map $G^{\mathbb{N}} \to X$.*

- *For ν-almost every $\xi \in X$, the* conditional entropy of ρ given ξ *is*

$$H_\xi(\rho) = -\sum_{k\geq 1} \mathbb{P}^\xi[\rho_k] \log \mathbb{P}^\xi[\rho_k].$$

- *The* mean conditional entropy of ρ over the μ-boundary $\mathbf{X}$ *is*

$$H_{\mathbf{X}}(\rho) = \int_X H_\xi(\rho) \, \mathrm{d}\nu(\xi).$$

- *The* asymptotic conditional entropy of ρ over the μ-boundary $\mathbf{X}$ *is*

$$h(\mu \mid \mathbf{X}) = \lim_{n\to\infty} H_{\mathbf{X}}(\alpha_n)/n.$$

We now formulate Kaimanovich's conditional entropy criterion in terms of the mean conditional entropy, which is our main tool in the proof of Theorem F.

Theorem 4.2.5 ([Kai85], [Kai00, Theorem 4.6])**.** *Assume that $H(\mu) < \infty$ and consider a μ-boundary $\mathbf{X}$ of G. Then $\mathbf{X}$ is the Poisson boundary of (G, μ) if and only if $h(\mu \mid \mathbf{X}) = 0$.*

For the equivalence between Theorem 4.2.5 and the original formulations of the conditional entropy criterion we refer to the explanation following Theorem 2.4 in [FS23].

In the proof of Theorem F it will be convenient to modify the step distribution μ in the following two ways.

Lemma 4.2.6. *Let μ be a non-degenerate probability measure on a countable group G. Consider a probability measure $\widetilde{\mu}$ equal either to*

- $\mu_{\text{lazy}} = \frac{1}{2}\mu + \frac{1}{2}\delta_{e_G}$*, or to*
- *a convolution μ^{*s} for some $s \in \mathbb{N}_+$.*

Then $H(\mu) < \infty$ if and only if $H(\widetilde{\mu}) < \infty$, and the Poisson boundary of (G, μ) is G-equivariantly measurably isomorphic to the Poisson boundary of $(G, \widetilde{\mu})$.

Proof. The fact that $H(\mu) < \infty$ if and only if $H(\widetilde{\mu}) < \infty$ follows from a direct computation. To see that the Poisson boundaries do not change, note that μ_{lazy} (resp. μ^{*s}) can be obtained from μ by stopping the random walk driven by μ along the stopping time $\tau = \inf\{k \geq 1 : g_k \neq 0\}$ (resp. $\tau = s$). The equality of the Poisson boundaries then follows from [For15, Theorem 3.6.1]. □

4.2.4 Groups acting on the circle

We refer to the monographs [Ghy01, Nav11] for general references on groups that act on the circle. For an overview on group actions on 1-manifolds and related topics, we refer to the surveys [Nav18, Man23].

A continuous surjection $\pi\colon S^1 \to S^1$ is said to have *degree* $d \in \mathbb{N}_+$ if any lift $\widetilde{\pi}\colon \mathbb{R} \to \mathbb{R}$ of π satisfies $\widetilde{\pi}(x + d) = \widetilde{\pi}(x) + d$ for all $x \in \mathbb{R}$. We say that an action $G \curvearrowright^{\phi} S^1$ is *semiconjugate* to $G \curvearrowright^{\psi} S^1$ if there exists a continuous surjection $\pi\colon S^1 \to S^1$ with $\pi \circ \phi(g) = \psi(g) \circ \pi$ for all $g \in G$, and such that π is locally non-decreasing and has degree one.

Recall that we call an action $G \curvearrowright S^1$ *minimal* if all orbits are dense in S^1. A folklore theorem describing the topological dynamics of an arbitrary group action asserts that unless $G \curvearrowright S^1$ has a finite orbit, for most purposes it suffices to consider minimal actions.

Theorem 4.2.7 (see [Ghy01, Proposition 5.6])**.** *Consider a group action $G \curvearrowright^{\phi} S^1$ by orientation-preserving homeomorphisms. Then exactly one of the following statements is satisfied.*

i. There exists a finite orbit.

ii. There exists a unique closed minimal set Λ, which is either S^1 or a Cantor set. In the latter case, by collapsing the connected componenents of $S^1 \setminus \Lambda$ we can semiconjugate ϕ to a minimal group action of G on S^1.

Consider a group action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms. The action is said to be *locally proximal* if there exists $r > 0$ such that for every interval $I \subseteq S^1$ with $\mathrm{diam}(I) < r$ and every $\varepsilon > 0$ there is $g \in G$ with $\mathrm{diam}(g(I)) < \varepsilon$. The action is said to be *proximal* if the previous is true for all closed intervals I strictly contained in S^1.

A complementary description of the dynamics of a minimal group action on S^1, in the spirit of the Tits alternative, is the following theorem by G. Margulis, which also follows from results of V. Antonov.

Theorem 4.2.8 ([Mar00, Ant84], see [Ghy01, Section 5.2]). *Consider a minimal group action $G \curvearrowright^{\phi} S^1$ by orientation-preserving homeomorphisms. Then exactly one of the following statements is satisfied.*

i. the action is conjugated to a minimal action by rotations, or

ii. the action is proximal, or

iii. the action is locally proximal and not proximal, and there exists $d \in \mathbb{N}$, $d \geq 2$ and a continuous d-to-one covering $\pi\colon S^1 \to S^1$ that intertwines ϕ with a proximal action.

As a consequence we have a dichotomy for group actions $G \curvearrowright S^1$: either

- the action preserves a probability measure on S^1, and this happens exactly when there is a finite G-orbit in S^1 or when $G \curvearrowright S^1$ is semiconjugate to a minimal action by rotations, or
- there exists $d \in \mathbb{N}_+$ and a continuous surjection $\pi\colon S^1 \to S^1$ that intertwines $G \curvearrowright S^1$ with a minimal and proximal action, such that the fibers $\pi^{-1}(x)$ are of size d for Lebesgue-almost every $x \in S^1$.

Denote by Leb the Lebesgue measure on $S^1 \cong \mathbb{R}/\mathbb{Z}$. The next theorem, due to Deroin-Kleptsyn-Navas, shows how μ-boundaries arise in S^1 for random walks on groups acting proximally. Recall that μ is a non-degenerate probability measure on G.

Theorem 4.2.9 ([DKN07, Appendice]). *Consider a group action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1.*

i. There exists a unique μ-stationary probability measure ν on S^1, which is atomless and is supported on the minimal set of G.

ii. If the action of G on S^1 is proximal, there exists a random variable $\mathbf{w} \in G^{\mathbb{N}} \mapsto \xi(\mathbf{w}) \in S^1$ such that for $\mathbb{P}$-almost every $\mathbf{w} = (w_n)_{n\geq 0}$ we have

$$(w_n)_*\mathrm{Leb} \xrightarrow[n\to\infty]{} \delta_{\xi(\mathbf{w})}$$

in the weak-$*$ topology.

In particular, $\lim_{n\to\infty}(w_n)_*\nu = \delta_{\xi(\mathbf{w})}$ holds $\mathbb{P}$-almost surely, and hence the circle (S^1, ν) is a μ-boundary of G.

Given $g \in \mathrm{Homeo}_0(S^1)$, denote by $\mathrm{supp}(g)$ the closure of the set $\{x \in S^1 : g(x) \neq x\}$. Recall that an action $G \curvearrowright S^1$ is *topologically non-free* if there exists an element $g \in G$ such that $\mathrm{supp}(g)$ is non-empty and is not all of S^1.

The proof of Theorem 4.2.8 shows that, whenever G is a group acting minimally and proximally on S^1, any open subset of S^1 can be contracted into any non-empty open subset of S^1 under the action of G (which is *a priori* stronger than the separate properties of minimality and proximality of the action). For the convenience of the reader, we present here a probabilistic proof of this result.

Proposition 4.2.10. *Consider a minimal and proximal group action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms. Then, for any pair of non-empty closed intervals $I, J \subsetneq S^1$ with non-empty interior there exists $g \in G$ such that $g(I) \subseteq J$. If the action $G \curvearrowright S^1$ is furthermore topologically non-free, then for every non-trivial interval $J \subseteq S^1$ there exists $a \in G \setminus \{e_G\}$ such that* $\mathrm{supp}(a) \subseteq J$.

Proof. Let I, J be non-empty closed proper intervals of S^1 with non-empty interior, and consider the right random walk $(w_n)_{n\geq 0} \in G^{\mathbb{N}}$ driven by μ. Item (ii) of Theorem 4.2.9 states that for $\mathbb{P}$-almost every trajectory $\mathbf{w} = (w_n)_{n\geq 0} \in G^{\mathbb{N}}$ and for any closed interval $K \subseteq S^1$ that does not contain $\xi(\mathbf{w})$, we have $\lim_{n\to\infty} \mathrm{diam}(w_n^{-1}(K)) = 0$. The distribution ν of $\xi(\mathbf{w})$ has $\mathrm{supp}(\nu) = S^1$, and therefore with positive probability $\xi(\mathbf{w}) \notin I$. From this, we obtain $\lim_{n\to\infty} \mathrm{diam}(w_n^{-1}(I)) = 0$. However, since $G \curvearrowright S^1$ is minimal and μ is non-degenerate, we have that $\mathbb{P}$-almost surely for any $x \in S^1$ the orbit $(w_n^{-1}(x))_{n\geq 0}$ is dense in S^1 [Fur02, Theorem 3.3]. In particular, when x is the left endpoint of I, the above implies that $\mathbf{w} \in G^{\mathbb{N}}$ and $n \in \mathbb{N}$ such that $w_n^{-1}(I) \subseteq J$. This shows the first statement of the proposition.

Now suppose that the action of G on S^1 is topologically non-free, so there is $b \in G \setminus \{e_G\}$ such that $\mathrm{supp}(b) \neq S^1$ and we may choose a closed interval $I \subsetneq S^1$ that contains $\mathrm{supp}(b)$. By the first statement of the proposition we can find $g \in G$ such that $g(I) \subseteq J$. Then the element $a = gbg^{-1} \in G \setminus \{e_G\}$ satisfies $\mathrm{supp}(a) \subseteq J$. This shows the second statement of the proposition. $\square$

4.3 Lower bounds on entropy

The aim of this section is to prove Theorem 4.3.2, which is a general criterion to show that mean conditional asymptotic entropy is positive and consists of a conditional version of the method used by A. Erschler [Ers04] to show that the asymptotic entropy $h(\mu)$ of a random walk is positive. Subsection 4.3.1 introduces Erschler's method and useful notation, and Subsection 4.3.2 proves Theorem 4.3.2. In this section, G is a countable group and μ is a non-degenerate probability measure on G.

We recall the method of [Ers04] exclusively for expositional purposes since this result will not be used in the proofs of our theorems. Nonetheless, we believe that understanding the statement of Theorem [Ers04, Theorem 2.1] (see Theorem 4.3.1 below) is useful for the understanding of its conditional version that we state below (Theorem 4.3.2), and which is key for the proof of Theorem F.

4.3.1 Estimating asymptotic entropy

We first recall some definitions from [Ers04]. A *collection of length* $n \in \mathbb{N}$ is a tuple $q = (\mathbf{x}, i_1, \ldots, i_k)$ where $0 < i_1 < \cdots < i_k \leq n$ are integers and

$$\mathbf{x} = (x_1, x_2, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{i_2-1}, x_{i_2+1}, \ldots, x_{i_k-1}, x_{i_k+1}, \ldots, x_n)$$

is an $(n-k)$-tuple of elements of $\mathrm{supp}(\mu)$. Notice that we index the elements of $\mathbf{x}$ by integers in $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$.

For each $a, b \in \mathrm{supp}(\mu)$ and $q = (\mathbf{x}, i_1, \ldots, i_k)$ a collection of length n, define $T^{a,b}(q)$ as the *set of trajectories* $\mathbf{y} = (y_1, \ldots, y_n) \in G^n$ such that, after setting $y_0 = 1$, $i_0 = 0$, we have

- for all $l \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ we have $y_l = y_{l-1}x_l$, and
- for all $l \in \{i_1, \ldots, i_l\}$, we have $y_l = y_{l-1}a$ or $y_l = y_{l-1}b$.

Thus, $T^{a,b}(q)$ is a set of 2^k trajectories of the μ-random walk on G up to time n. We say that $T^{a,b}(q)$ is *satisfactory* if all trajectories in $T^{a,b}(q)$ arrive at different elements of G at time n. That is, if $(y_1, \ldots, y_n), (y'_1, \ldots, y'_n)$ are distinct trajectories in $T^{a,b}(q)$, then $y_n \neq y'_n$.

For a trajectory $\mathbf{y} = (y_1, \ldots, y_n)$, we define the *jumps* of $\mathbf{y}$ as $g_j = y_{j-1}^{-1}y_j \in \mathrm{supp}(\mu)$ for all $1 \leq j \leq n$, and we denote by

$$[\mathbf{y}] = \{\mathbf{w} \in G^{\mathbb{N}} : w_j = y_j \text{ for all } 1 \leq j \leq n\}$$

the cylinder defined by $\mathbf{y}$.

Theorem 4.3.1 ([Ers04])**.** *Assume that $H(\mu) < \infty$. Suppose that there exist $p, c > 0$ such that for each $n \in \mathbb{N}_+$ there is a set A_n of collections of length n that verify the following conditions.*

i. For each $q = (\mathbf{x}, i_1, \ldots, i_k) \in A_n$ we have $k \geq cn$.

ii. For each $q \in A_n$ the set of trajectories $T^{a,e_G}(q)$ is satisfactory.

iii. For each $q_1, q_2 \in A_n$ with $q_1 \neq q_2$ we have $[\mathbf{y}_1] \cap [\mathbf{y}_2] = \varnothing$ whenever $\mathbf{y}_1 \in T^{a,e_G}(q_1)$ and $\mathbf{y}_2 \in T^{a,e_G}(q_2)$.

iv. We have

$$\mathbb{P}\left[\bigcup_{q \in A_n} \bigcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}]\right] \geq p.$$

Then the asymptotic entropy $h(\mu)$ is positive.

To gain some intuition, here is a vague rephrasing of the assumptions. With positive probability, the random walk trajectory is uniquely assigned a linear number of distinguished times along the trajectory together with fixed steps for all other time instants. At any of these distinguished times, the choice between doing an increment of e_G or of a will lead the trajectory to different endpoints, regardless of which choices are made at later distinguished instants.

4.3.2 Estimating conditional asymptotic entropy

We have the following conditional version of Theorem 4.3.1, which has a similar structure to Theorem 4.3.1 but where an additional assumption is needed to handle the transition probabilities conditional to a boundary point.

Theorem 4.3.2. *Let μ be a probability measure on a countable group G with $H(\mu) < \infty$ and let $\mathbf{X} = (X, \nu)$ be a μ-boundary of G. Consider an element $a \in G \setminus \{e_G\}$ and a measurable subset $J \subseteq X$ such that $a(x) = x$ for each $x \in X \backslash J$. Suppose that there exist $p, c > 0$, for each $n \in \mathbb{N}_+$ a set $\Xi_n \subseteq X$ of measure $\nu(\Xi_n) \geq p$, and for ν-almost every $\xi \in \Xi_n$ a set $A_{n,\xi}$ of collections of length n that verify the following conditions.*

i. For each $q = (\mathbf{x}, i_1, \ldots, i_k) \in A_{n,\xi}$ we have $k \geq cn$.

ii. For each $q \in A_{n,\xi}$ the set of trajectories $T^{a,e_G}(q)$ is satisfactory.

iii. For each $q_1, q_2 \in A_{n,\xi}$ with $q_1 \neq q_2$ we have $[\mathbf{y}_1] \cap [\mathbf{y}_2] = \varnothing$ whenever $\mathbf{y}_1 \in T^{a,e_G}(q_1)$ and $\mathbf{y}_2 \in T^{a,e_G}(q_2)$.

iv. We have

$$\mathbb{P}^{\xi}\left[\bigcup_{q \in A_{n,\xi}} \bigcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}]\right] \geq p.$$

v. For each $q = (\mathbf{x}, i_1, \ldots, i_k) \in A_{n,\xi}$ and any $(y_1, \cdots, y_n) \in T^{a,e_G}(q)$ we have

$$y_{i_r - 1}^{-1}(\xi) \notin J \text{ for all } 1 \leq r \leq k.$$

Then the asymptotic conditional entropy $h(\mu \mid X) = \lim_{n\to\infty} H_{\mathbf{X}}(\alpha_n)/n$ is positive.

Proof. Notice that Condition (iii) implies that all the sets $A_{n,\xi}$ are non-empty, and hence the definition of $T^{a,e_G}(q)$ for a collection q and Condition (i) imply that a and e_G belong to the support of μ.

Let us fix the notation that we will use in the rest of the proof. Given a collection q, denote by $Q^{a,e_G}(q) \subseteq G^{\mathbb{N}}$ the union $\bigcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}]$. Fix $n \in \mathbb{N}$, and consider $\xi \in X$, a countable measurable partition η of $G^{\mathbb{N}}$ and q a collection of length n such that $\mathbb{P}^{\xi}[Q^{a,e_G}(q)] > 0$. Denote by $H^{\xi}(\eta, q)$ the entropy of the partition

$$\{P \cap Q^{a,e_G}(q) : P \in \eta\}$$

of $Q^{a,e_G}(q)$ with respect to the normalized probability measure $\mathbb{P}^{\xi}/\mathbb{P}^{\xi}\left[Q^{a,e_G}(q)\right]$ restricted to $Q^{a,e_G}(q)$. If $\mathbb{P}^{\xi}\left[Q^{a,e_G}(q)\right] = 0$, set $H^{\xi}(\eta, q) = 0$.

Let α_n be the partition of $G^{\mathbb{N}}$ where two trajectories belong to the same atom of α_n if and only if they hit the same element of the group at time n. Then we have

$$H_{\mathbf{X}}(\alpha_n) = \int_X H^{\xi}(\alpha_n)\, \mathrm{d}\nu(\xi) \geq \int_X \sum_{q \in A_{n,\xi}} H^{\xi}(\alpha_n, q)\mathbb{P}^{\xi}\left[Q^{a,e_G}(q)\right] \mathrm{d}\nu(\xi), \tag{4.3.1}$$

where in the last inequality we used the fact that the sets $Q^{a,e_G}(q)$ are disjoint by Condition (iii).

By Condition (ii), the partition $\{P \cap Q^{a,e_G}(q) : P \in \alpha_n\}$ coincides with the partition

$$\{\,[\mathbf{y}] : \mathbf{y} \in T^{a,e_G}(q)\}.$$

Hence, by Lemma 4.2.3 and Condition (v), we obtain that for any $\mathbf{y} \in T^{a,e_G}(q)$ with increments $g_1, \ldots, g_n$ we have

$$\begin{aligned}
\mathbb{P}^\xi\,[[\mathbf{y}]] &= \prod_{j=1}^{n} \mathbb{P}^\xi\,[w_j = g_1 \cdots g_{j-1} g_j \mid w_{j-1} = g_1 \cdots g_{j-1}] \\
&= \prod_{r=1}^{k} \mu(g_{i_r}) \prod_{\substack{j=1 \\ j \notin \{i_s\}_s}}^{n} \mathbb{P}^\xi\,[w_j = g_1 \cdots g_{j-1} g_j \mid w_{j-1} = g_1 \cdots g_{j-1}] \\
&= \prod_{r=1}^{k} \frac{\mu(g_{i_r})}{\mu(a) + \mu(e_G)} \prod_{r=1}^{k} \mathbb{P}^\xi\left[w_{i_r} w_{i_r-1}^{-1} \in \{a, e_G\} \mid w_{i_r-1}\right] \prod_{\substack{j=1 \\ j \notin \{i_s\}_s}}^{n} \mathbb{P}^\xi\left[w_j w_{j-1}^{-1} = x_j \mid w_{j-1}\right] \\
&= \prod_{r=1}^{k} \frac{\mu(g_{i_r})}{\mu(a) + \mu(e_G)} \mathbb{P}^\xi\,[Q^{a,e_G}(q)]\,.
\end{aligned}$$

We deduce that

$$\frac{\mathbb{P}^\xi[[\mathbf{y}]]}{\mathbb{P}^\xi\,[Q^{a,e_G}(q)]} = \frac{\mu(a)^{A(\mathbf{y})} \mu(e_G)^{k-A(\mathbf{y})}}{(\mu(a) + \mu(e_G))^k},$$

where $A(\mathbf{y}) \in \mathbb{N}$ is the number of times that a appears in the sequence $(g_{i_r})_{r=1}^k$. Denote by ρ the Bernoulli measure on $\{a, e_G\}$ giving weight $\frac{\mu(a)}{\mu(e_G)+\mu(a)}$ to a and $\frac{\mu(e_G)}{\mu(e_G)+\mu(a)}$ to e_G. We conclude that $H^\xi(\alpha_n, q) = H(\rho^{k(q)}) = k(q)H(\rho)$. Since $\mu(a)$ and $\mu(e_G)$ are positive, the quantity $H(\rho)$ is also positive.

We have thus from Condition (i) that $H^\xi(\alpha_n, q) = k(q)H(\rho) \geq cnH(\rho)$, and therefore

$$\begin{aligned}
\int_{\Xi_n} \sum_{q \in A_{n,\xi}} H^\xi(\alpha_n, q)\mathbb{P}^\xi\,[Q^{a,e_G}(q)]\,\mathrm{d}\nu(\xi) &\geq cnH(\rho) \int_{\Xi_n} \mathbb{P}^\xi \left[\bigcup_{q \in A_{n,\xi}} Q^{a,e_G}(q)\right] \mathrm{d}\nu(\xi) \\
&\geq cnH(\rho)p^2.
\end{aligned}$$

Finally, from Equation (4.3.1) we deduce that $H_{\mathbf{X}}(\alpha) \geq cH(\rho)p^2 n$ for any $n \in \mathbb{N}$. This shows that $h(\mu \mid \mathbf{X}) \geq cH(\rho)p^2 > 0$ and finishes the proof. □

4.4 Good collections

The purpose of this section is to prove Lemmas 4.4.2 and Proposition 4.4.5, which give sufficient conditions to verify the hypotheses of Theorem 4.3.2. In this section, μ be a non-degenerate probability measure on a countable subgroup G of $\mathrm{Homeo}_0(S^1)$ acting minimally and proximally on S^1.

Let $a \in G \setminus \{e_G\}$ be an element such that $S^1 \setminus \mathrm{supp}(a)$ has non-empty interior, and let $J \subsetneq S^1$ be the smallest closed interval containing $\mathrm{supp}(a)$.

Definition 4.4.1. *Given two closed intervals $I_1, I_2 \subseteq S^1$, we say that I_1* dominates *I_2 if they are disjoint or if the interior of I_1 contains I_2.*

Lemma 4.4.2. *Fix a collection $q = (\mathbf{x}, i_1, \dots, i_k)$ and consider two trajectories of length n given by $(y_1, \dots, y_n), (\widetilde{y}_1, \dots, \widetilde{y}_n) \in T^{a,e_G}(q)$.*

Then, for each $1 \leq r \leq k$ the following statements are equivalent.

i. For all $0 \leq l < i_r - 1$, the interval $y_{i_r-1}(J)$ dominates $y_l(J)$.

ii. For all $0 \leq l < i_r - 1$, the interval $\widetilde{y}_{i_r-1}(J)$ dominates $\widetilde{y}_l(J)$.

Whenever these conditions are met, we have $y_{i_r-1}(J) = \widetilde{y}_{i_r-1}(J)$ for all $0 \leq r \leq k$ and that q is satisfactory.

Proof. By symmetry, to prove the first statement it suffices to see that Condition (i) implies Condition (ii). Denote by $g_1, \dots, g_n$ and $\widetilde{g}_1, \dots, \widetilde{g}_n$ the jumps of $(y_1, \dots, y_n)$ and $(\widetilde{y}_1, \dots, \widetilde{y}_n)$ respectively. We have $i_{r-1} < i_r - 1$ for every $1 \leq r \leq k$: indeed, if $i_{r-1} = i_r - 1$ then $y_{i_r-2}^{-1} y_{i_r}(J) = g_{i_{r-1}} g_{i_r}(J) = J$ so $y_{i_r}(J) = y_{i_r-2}(J)$, contradicting Condition (i).

Fix $1 \leq r \leq k$. We will prove by backwards induction on $l = i_r - 2, i_r - 3, \dots, 0$ that

$$\widetilde{y}_l^{-1} \widetilde{y}_{i_r-1}(J) \text{ dominates } J \quad \text{and} \quad \widetilde{y}_l^{-1} \widetilde{y}_{i_r-1}(J) = y_l^{-1} y_{i_r-1}(J). \tag{H_l}$$

For the base case $l = i_r - 2$, notice that since $i_r - 1 \neq i_{r-1}$, we have $\widetilde{y}_l^{-1} \widetilde{y}_{i_r-1} = g_{i_r-1} = y_l^{-1} y_{i_r-1}$, showing (H_{i_r-2}).

Now take $0 \leq l < i_r - 2$ and assume (H_{l+1}).

If $l = i_k$ for some $0 \leq k \leq r$, then (H_{l+1}) implies that $\widetilde{y}_{l+1}^{-1} y_{i_r-1}(J)$ is either disjoint or contains J, the support of $\widetilde{g}_l$. Hence

$$\widetilde{y}_l^{-1} y_{i_r-1}(J) = \widetilde{g}_l \widetilde{y}_{l+1}^{-1} y_{i_r-1}(J) = \widetilde{y}_{l+1}^{-1} y_{i_r-1}(J), \tag{4.4.1}$$

and we deduce that

$$\widetilde{y}_l^{-1} \widetilde{y}_{i_r-1}(J) = \widetilde{g}_l \widetilde{y}_{l+1}^{-1} \widetilde{y}_{i_r-1}(J) \text{ dominates } J = \widetilde{g}_l(J) \tag{4.4.2}$$

since (H_{l+1}) holds. Moreover, by Condition (i) the interval $y_{l+1}^{-1} y_{i_r-1}(J)$ is either disjoint or contains J, the support of g_l, so

$$y_l^{-1} y_{i_r-1}(J) = g_l y_{l+1}^{-1} y_{i_r-1}(J) = y_{l+1}^{-1} y_{i_r-1}(J)$$

too, which together with Equation (4.4.1) gives

$$y_l^{-1} y_{i_r-1}(J) = y_{l+1}^{-1} y_{i_r-1}(J) = \widetilde{y}_{l+1}^{-1} y_{i_r-1}(J) = \widetilde{y}_k^{-1} y_{i_r-1}(J).$$

The previous equation and Equation (4.4.2) give (H_l).

If $l \neq i_k$ for all $0 \leq k \leq r$ instead, then

$$\widetilde{y}_l^{-1}\widetilde{y}_{i_r-1}(J) = g_l\widetilde{y}_{l+1}^{-1}\widetilde{y}_{i_r-1}(J) = g_l y_{l+1}^{-1} y_{i_r-1}(J) = y_l^{-1} y_{i_r-1}(J)$$

and hence $\widetilde{y}_l^{-1}\widetilde{y}_{i_r-1}(J) = y_l^{-1}y_{i_r-1}(J)$ dominates J. This shows (H_l), finishing the induction and the proof of Condition (ii).

To show the second statement, it remains to show that the collection q is satisfactory if there exists a collection in $T^{a,e_G}(q)$ satisfying Condition (i). To show this, take two distinct trajectories $(y_1,\dots,y_n)$ and $(\widetilde{y}_1,\dots,\widetilde{y}_n)$ in $T^{a,e_G}(q)$ with jumps $g_{i_1},\dots,g_{i_r}$ and $\widetilde{g}_{i_1},\dots,\widetilde{g}_{i_r} \in \{a, e_G\}$ respectively at times $i_1,\dots,i_k$. For $1 \leq r \leq k$ write $e_r = \widetilde{g}_{i_r} g_{i_r}^{-1} \in G$. Denote by $b_1,\dots,b_k \in G$ the blocks of q between the times $i_1,\dots,i_k$, so $y_n = b_1 g_{i_1} b_2 g_{i_2} \cdots b_k g_{i_k} b_{k+1}$ and $\widetilde{y}_n = b_1\widetilde{g}_{i_1} b_2 \widetilde{g}_{i_2}\cdots b_k \widetilde{g}_{i_k} b_{k+1}$.

Write

$$\begin{aligned}
\widetilde{y}_n y_n^{-1} &= b_1\widetilde{g}_{i_1}\cdots\widetilde{g}_{i_{k-1}} b_k e_k b_k^{-1} g_{i_{k-1}}^{-1}\cdots g_{i_1}^{-1} b_1^{-1}\\
&= b_1\widetilde{g}_{i_1}\cdots\widetilde{g}_{i_{k-2}} b_{k-1} e_{k-1} e_k^{g_{i_{k-1}} b_k} b_{k-1}^{-1} g_{i_{k-2}}^{-1}\cdots g_{i_1}^{-1} b_1^{-1}\\
&= b_1\widetilde{g}_{i_1}\cdots\widetilde{g}_{i_{k-2}} b_{k-1} e_{k-1} b_{k-1}^{-1} g_{i_{k-2}}^{-1}\cdots g_{i_1}^{-1} b_1^{-1} e_k^{b_1 g_{i_1}\cdots g_{i_{k-1}} b_k}
\end{aligned}$$

where we have used the notation $u^v = vuv^{-1}$. By iterating the previous calculation we arrive at

$$\widetilde{y}_n y_n^{-1} = e_1^{b_1} e_2^{b_1 g_{i_1} b_2} \cdots e_{k-1}^{b_1 g_{i_1}\cdots g_{i_{k-2}} b_{k-1}} e_k^{b_1 g_{i_1}\cdots g_{i_{k-1}} b_k} = e_1^{y_{i_1-1}} e_2^{y_{i_2-1}} \cdots e_{k-1}^{y_{i_{k-1}-1}} e_k^{y_{i_k-1}}.$$

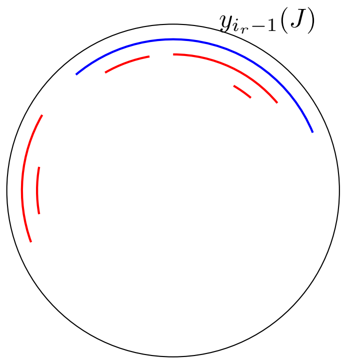

Figure 4.1. All red intervals $y_{i_{r'}-1}(J)$ are dominated by the blue interval $y_{i_r-1}(J)$.

Notice that the support of $e_r^{y_{i_r-1}}$ is $y_{i_r-1}(\mathrm{supp}(a))$. Choose $1 \leq r \leq k$ such that $e_r \neq 1$ and for all $1 \leq r' \leq k$ with $e_{r'} \neq 1$ either $y_{i_{r'}-1}(J)$ is strictly contained in $y_{i_r-1}(J)$ or is disjoint from $y_{i_r-1}(J)$. Then $\widetilde{y}_n y_n^{-1}$ coincides in a neighborhood of $\partial y_{i_r-1}(J)$ with $e_r^{y_{i_r-1}}$ (see Figure 4.1). This neighborhood must intersect $y_{i_r-1}(\mathrm{supp}(a))$ because J is the smallest subinterval containing $\mathrm{supp}(a)$, and hence $\widetilde{y}_n y_n^{-1}$ is non-trivial near $\partial y_{i_r-1}(J)$. We conclude that q is satisfactory. □

Definition 4.4.3. *Given $\xi \in S^1$, we say that $q = (\mathbf{x}, i_1, \dots, i_k)$ is* ξ-good *if*

- *for all trajectories* $(y_1, \dots, y_n) \in T^{a,e_G}(q)$*, every* $1 \le r \le k$*, we have*

$$y_{i_r-1}(J) \text{ dominates } y_l(J) \text{ for all } 0 \le l < i_r - 1 \quad \text{and} \quad y_{i_r}^{-1}(\xi) \notin J, \text{ and}$$

- *the set of indices* $i_1, \dots, i_k$ *is maximal among the subsets of* $\{1, \dots, n\}$ *satisfying the property above.*

Equivalently, by Lemma 4.4.2 the collection q is ξ-good if there exists at least one trajectory in $T^{a,e_G}(q)$ that verifies the previous conditions.

Lemma 4.4.4. *For every* $n \in \mathbb{N}$ *and* $\xi \in S^1$ *the set of trajectories is partitioned as*

$$G^{\mathbb{N}} = \bigsqcup_{\substack{q \text{ a } \xi\text{-good} \\ \text{collection of length } n}} \ \bigsqcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}].$$

Proof. Given any trajectory $\mathbf{w} = (w_n)_{n \ge 0}$ there is exactly one ξ-good collection q such that $\mathbf{w}$ is contained in $\bigcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}]$, which is defined by setting $q = (\mathbf{x}, i_1, \dots, i_k)$ where the i_r are exactly the indices such that

- for all $0 \le l < i_r - 1$ the interval $w_{i_r-1}(J)$ dominates $w_l(J)$ and
- $w_{i_r}^{-1}(\xi) \notin J$,

and $\mathbf{x}$ is obtained from $\mathbf{w}$ by keeping the coordinates in $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$. □

Notice that whenever $A_{n,\xi}$ is composed of ξ-good collections, Lemma 4.4.2 implies that Conditions (ii) and (v) in Theorem 4.3.2 are immediately satisfied. Moreover, the previous lemma ensures that Condition (iii) in the theorem is also verified. We record this as a proposition, which we will use to verify some of the hypotheses of Theorem 4.3.2.

Proposition 4.4.5. *Let* μ *be a non-degenerate probability measure with finite entropy on a countable subgroup* G *of* $\mathrm{Homeo}_0(S^1)$ *acting minimally and proximally on* S^1*. Consider the* μ*-boundary* (S^1, ν) *of* G*. Let* $a \in G \setminus \{e_G\}$ *be an element such that both* $\mathrm{supp}(a)$ *and* $S^1 \setminus \mathrm{supp}(a)$ *have non-empty interior, and let* $J \subsetneq S^1$ *be the smallest closed interval containing* $\mathrm{supp}(a)$*. Let* $p > 0$ *and consider for each* $n \in \mathbb{N}_+$ *a subset* $\Xi_n \subseteq S^1$ *with* $\nu(\Xi_n) \ge p$*, and for* ν*-almost every* $\xi \in \Xi_n$ *a set* $A_{n,\xi}$ *of collections of length* n*. If for every* $n \in \mathbb{N}$ *and* $\xi \in \Xi_n$ *the set* $A_{n,\xi}$ *is composed of* ξ*-good collections, then Conditions* (ii)*,* (iii) *and* (v) *in Theorem 4.3.2 are satisfied.*

4.5 Exponential contraction in mean

This section is to prove Corollary 4.5.2 and Proposition 4.5.3 below, which are the remaining statements on random walks on $\mathrm{Homeo}_0(S^1)$ that we need to prove Proposition 4.6.2 and hence Theorem F. In this section μ is a non-degenerate probability measure on $\mathrm{Homeo}_0(S^1)$ acting minimally and proximally on S^1.

The following theorem has already appeared in the literature in several guises, see [Aou11, Proposition 4.15] for probability measures satisfying an exponential moment condition on linear groups acting on projective spaces and [GS23, Theorem 1.3] or [GK21, Proposition 4.18] for finitely supported measures on $\mathrm{Diff}^1_0(S^1)$. The most general version follows from the recent work of I. Choi [Cho25], and does not require any assumption on the smoothness of the elements of G nor on the tail decay of the probability measure μ.

Theorem 4.5.1 ([Cho25])**.** *There exists $\lambda > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

$$\sup_{x,y\in S^1} \mathbb{E}\left[d(w_n^{-1}(x), w_n^{-1}(y))\right] \leq e^{-\lambda n}.$$

Proof. It follows from [Cho25, Theorem C] that there exists $\lambda > 1$ such that for all $x, y \in S^1$ and $n \in \mathbb{N}_+$ we have

$$\mathbb{P}\left[d(w_n^{-1}(x), w_n^{-1}(y)) \leq e^{-\lambda n}\right] \geq 1 - e^{-\lambda n}/\lambda.$$

From this, we obtain

$$\mathbb{E}\left[d(w_n^{-1}(x), w_n^{-1}(y))\right] \leq (1 + 1/\lambda)e^{-\lambda n},$$

which implies the desired inequality. □

The proof of the following corollary uses Theorem 4.5.1 and follows steps similar to the proof of [Aou11, Theorem 4.16].

Corollary 4.5.2 (Exponential convergence in mean to the boundary point)**.** *Let μ be a non-degenerate probability measure on a countable subgroup G of $\mathrm{Homeo}_0(S^1)$ acting minimally and proximally on S^1. Denote by $\xi \colon G^{\mathbb{N}} \to S^1$ the boundary map. Then there exist $\lambda > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

$$\sup_{x\in S^1} \mathbb{E}[d(w_n(x), \xi(\mathbf{w}))] \leq e^{-\lambda n}.$$

Proof. Let $n, k \in \mathbb{N}_+$ with $n < k$ and let $x, y \in S^1$. We have that

$$\mathbb{E}\left[d\left(w_n(x), \xi(\mathbf{w})\right)\right] \leq \mathbb{E}\left[d\left(w_n(x), w_k(y)\right)\right] + \mathbb{E}\left[d\left((w_k(y), \xi(\mathbf{w})\right)\right].$$

Define a probability measure $\overline{\mu}$ on G as $\overline{\mu}(g) = \mu(g^{-1})$ for $g \in G$. Theorem 4.5.1 applied to the random walk driven by $\overline{\mu}$ gives a $\lambda > 0$ such that

$$\sup_{u,v\in S^1} \mathbb{E}\left[d\left(g_n \cdots g_0(u), g_n \cdots g_0(v)\right)\right] \leq e^{-\lambda n}$$

for all large enough $n \in \mathbb{N}_+$. In particular, we deduce that

$$\begin{aligned}
\mathbb{E}\left[d\left(w_n(x), w_k(y)\right)\right] &= \sum_{\gamma\in G} \mathbb{E}\left[d\left(w_n(x), w_n\gamma(y)\right)\right] \mu^{*(k-n)}(\gamma) \\
&\leq \sup_{u,v\in S^1} \mathbb{E}\left[d\left(w_n(u), w_n(v)\right)\right] \\
&= \sup_{u,v\in S^1} \mathbb{E}\left[d\left(g_n \cdots g_0(u), g_n \cdots g_0(v)\right)\right] \leq e^{-\lambda n},
\end{aligned}$$

and hence conclude that

$$\sup_{x\in S^1} \mathbb{E}\left[d\left(w_n(x),\xi(\mathbf{w})\right)\right] \leq e^{-\lambda n} + \mathbb{E}\left[d\left(w_k(y),\xi(\mathbf{w})\right)\right].$$

By integrating the above inequality with respect to ν we conclude that

$$\begin{aligned}\sup_{x\in S^1} \mathbb{E}\left[d\left(w_n(x),\xi(\mathbf{w})\right)\right] &\leq e^{-\lambda n} + \mathbb{E}\left[\int_{S^1} d\left(w_k(y),\xi(\mathbf{w})\right)\mathrm{d}\nu(y)\right]\\ &= e^{-\lambda n} + \mathbb{E}\left[\int_{S^1} d(y,\xi(\mathbf{w}))\,\mathrm{d}w_k\nu(y)\right].\end{aligned}$$

But the dominated convergence theorem and Theorem 4.2.9, (ii) imply that

$$\mathbb{E}\left[\int_{S^1} d(y,\xi(\mathbf{w}))\,\mathrm{d}w_k\nu(y)\right] \xrightarrow[k\to\infty]{} \mathbb{E}\left[\int_{S^1} d(y,\xi(\mathbf{w}))\,\mathrm{d}\delta_{\xi(\mathbf{w})}(y)\right] = 0,$$

so the desired conclusion holds. □

For a probability measure μ on G, we denote by $\overline{\mu}$ the *reflected* probability measure on G, defined by $\overline{\mu}(g) = \mu(g^{-1})$ for each $g \in G$.

Proposition 4.5.3. *Let μ be a non-degenerate probability measure on a countable subgroup G of* $\mathrm{Homeo}_0(S^1)$ *acting minimally and proximally on S^1. Denote by $\xi\colon G^{\mathbb{N}} \to S^1$ the boundary map. Consider the reflected probability measure $\overline{\mu}$ on G, and denote by $\overline{\nu}$ the unique $\overline{\mu}$-stationary probability measure on S^1. Then for any non-empty interval $J \subseteq S^1$ there exists $N \in \mathbb{N}_+$ such that for all $n \geq N$,*

$$\mathbb{E}\left[\left|\{1 \leq k \leq n : \xi(\mathbf{w}) \in w_k(J)\}\right|\right] \leq 2\overline{\nu}(J)n.$$

Proof. Notice that $\overline{\nu}(J) > 0$ because J is non-empty and $\overline{\nu}$ has full support. Since

$$\mathbb{E}\left[\,|1 \leq k \leq n : \xi(\mathbf{w}) \in w_k(J)|\,\right] = \sum_{k=0}^{n} \mathbb{P}\left[\xi(\mathbf{w}) \in w_k(J)\right],$$

it suffices to prove that $\mathbb{P}\left[\xi \in w_n(J))\right] < 3\overline{\nu}(J)/2$ for all large enough $n \in \mathbb{N}_+$. Recall that we write $\sigma\colon G^{\mathbb{N}} \to G^{\mathbb{N}}$ for the shift map on the space of sample paths. Since the boundary map $\xi\colon (G^{\mathbb{N}},\mathbb{P}) \to (S^1,\nu)$ is σ-invariant, we have

$$\begin{aligned}\mathbb{P}\left[\xi(\mathbf{w}) \in w_n(J)\right] = \mathbb{P}\left[\xi(\sigma^n\mathbf{w}) \in w_n(J)\right] &= \sum_{g\in G} \mathbb{P}\left[\xi(\sigma^n\mathbf{w}) \in w_n(J) \mid w_n = g\right]\mu^{*n}(g)\\ &= \sum_{g\in G} \mathbb{P}\left[\xi(\sigma^n\mathbf{w}) \in g(J) \mid w_n = g\right]\mu^{*n}(g)\\ &= \sum_{g\in G} \mathbb{P}\left[\xi(\mathbf{w}) \in g(J)\right]\mu^{*n}(g)\\ &= \sum_{g\in G} \nu(g(J))\mu^{*n}(g). \qquad (4.5.1)\end{aligned}$$

If μ were to be symmetric, then ν would also be $\overline{\mu}$-stationary and we would conclude that $\mathbb{P}\left[\xi \in w_n(J)\right] = \nu(J)$, which implies the desired inequality. In the general case, when μ may not be symmetric, we proceed as follows. For every $\mathbf{w} = (w_n)_{n\geq 0} \in G^{\mathbb{N}}$ denote by $\overline{\xi}(\mathbf{w}) \in S^1$ the boundary point for the random walk $\{g_1^{-1}\cdots g_n^{-1}\}_{n\geq 0}$, so that $\mathbb{P}$-almost surely

$$(g_1^{-1}\cdots g_n^{-1})\overline{\nu} \xrightarrow[n\to\infty]{} \delta_{\overline{\xi}(\mathbf{w})}$$

in the weak-$*$ topology.

Since $\overline{\nu}$ has support equal to S^1, whenever $\overline{\xi}(\mathbf{w}) \notin J$ we have $\operatorname{diam}(g_n\cdots g_1(J)) \xrightarrow[n\to\infty]{} 0$. From this together with the fact that ν is non-atomic, we obtain

$$\nu(g_n\cdots g_1(J)) \xrightarrow[n\to\infty]{} 0. \tag{4.5.2}$$

Next, we have that

$$\begin{aligned}\sum_{g\in G}\nu(g(J))\mu^{*n}(g) &= \int_{G^{\mathbb{N}}}\nu(g_n\cdots g_1(J))\,\mathrm{d}\mathbb{P}(\mathbf{w})\\ &\leq \mathbb{P}\left[\overline{\xi}(\mathbf{w})\in J\right] + \int_{\overline{\xi}\notin J}\nu(g_n\cdots g_1(J))\,\mathrm{d}\mathbb{P}(\mathbf{w})\\ &= \overline{\nu}(J) + \int_{\overline{\xi}\notin J}\nu(g_n\cdots g_1(J))\,\mathrm{d}\mathbb{P}(\mathbf{w}). \qquad (4.5.3)\end{aligned}$$

The convergence of Equation (4.5.2) together with the dominated convergence theorem show that

$$\int_{\overline{\xi}\notin J}\nu(g_n\cdots g_1(J))\,\mathrm{d}\mathbb{P}(\mathbf{w}) \xrightarrow[n\to\infty]{} 0,$$

so the right side of Equation (4.5.3) is at most $3\overline{\nu}(J)/2$ for large enough $n \in \mathbb{N}$. Together with Equation (4.5.1), this proves the desired statement. □

4.6 There is a linear number of dominating intervals along the walk

The purpose of this section is to prove Proposition 4.6.2, the key to Theorem F. As before, in this section μ is a probability measure on a countable subgroup G of $\mathrm{Homeo}_0(S^1)$ acting minimally and proximally on S^1. Recall that we denote by $(w_n)_{n\geq 0}$ a sample path of the μ-random walk on G.

Definition 4.6.1. *For every $n, s \in \mathbb{N}_+$ with $1 < s < n$ and each proper interval $J \subseteq S^1$, define the random variable $Z_{n,s}^J \in \mathbb{N}$ as the number of times $1 \leq k \leq \lceil n/s\rceil$ such that the interval $w_{ks}(J)$ dominates $w_{js}(J)$ for all $0 \leq j \leq k-1$. That is, we set $Z_{n,s}^J = \sum_{k=1}^{\lceil n/s\rceil} 1_{B_k}$, where*

$$B_k = \{w_{ks}(J) \text{ dominates } w_{js}(J) \text{ for all } 0 \leq j \leq k-1\}$$

for every $1 \leq k \leq \lceil n/s\rceil$. We call the parameter $s \in \mathbb{N}_+$ the sparsity.

For every $l \in \mathbb{N}_+$ we denote by $\mathbb{P}_{\mu^{*l}}$ the probability measure on $G^{\mathbb{N}}$ given by the distribution of the trajectories of the μ^{*l}-random walk on G. Denote by $\mathbb{E}_{\mu^{*l}}$ the associated expectation. The following proposition guarantees that in expectation there is a linear number of dominated intervals along the trajectory of the random walk.

Proposition 4.6.2. *Let μ be a probability measure on a countable subgroup of* $\mathrm{Homeo}_0(S^1)$ *acting minimally and proximally on S^1, and denote by ν the unique μ-stationary probability measure on S^1. Let $I \subseteq S^1$ be a closed interval such that $\nu(I) < 1/2$ and let $J \subseteq I$. Then there exist $s, N \in \mathbb{N}_+$ and $0 < c < 1$ such that for all $l \in \mathbb{N}_+$ and every $n \geq N$ we have $\mathbb{E}_{\mu^{*l}}[Z^J_{n,s}] \geq cn$.*

The proof of this proposition will follow from the next two lemmas.

Lemma 4.6.3. *Consider the same hypotheses as in Proposition 4.6.2. Then for any $s, l \in \mathbb{N}_+$ there is $N \geq 1$ such that for all $n \geq N$ we have*

$$\mathbb{E}_{\mu^{*l}}[Z^J_{n,s}] \geq \frac{n}{2s}\mathbb{P}_{\mu^{*l}}\left[w_{js}(J) \text{ dominates } J \text{ for all } j \geq 1\right].$$

Proof. For $l \in \mathbb{N}_+$ we have

$$\begin{aligned}
\mathbb{E}_{\mu^{*l}}[Z^J_{n,s}] &= \sum_{k=1}^{\lceil n/s \rceil} \mathbb{P}_{\mu^{*l}}\left[w_{ks}(J) \text{ dominates } w_{js}(J) \text{ for all } 0 \leq j \leq k-1\right] \\
&= \sum_{k=1}^{\lceil n/s \rceil} \mathbb{P}_{\mu^{*l}}\left[g_{js+1}g_{js+2}\cdots g_{ks}(J) \text{ dominates } J \text{ for all } 0 \leq j \leq k-1\right] \\
&= \sum_{k=1}^{\lceil n/s \rceil} \mathbb{P}_{\mu^{*l}}\left[g_1 g_2 \cdots g_{(k-j)s}(J) \text{ dominates } J \text{ for all } 0 \leq j \leq k-1\right] \\
&= \sum_{k=1}^{\lceil n/s \rceil} \mathbb{P}_{\mu^{*l}}\left[w_{js}(J) \text{ dominates } J \text{ for all } 1 \leq j \leq k\right],
\end{aligned}$$

where the second to last equality follows from the fact that the increments $(g_j)_{j\geq 1}$ are independent and identically distributed.

For every $k = 1, 2, \ldots, \lceil n/s \rceil$ denote by D_k the event where $w_{js}(J)$ dominates J for all $1 \leq j \leq k$. Notice that $D_{k+1} \subseteq D_k$ for each $k = 1, 2, \ldots, \lceil n/s \rceil - 1$, and therefore we have that

$$\mathbb{P}_{\mu^{*l}}\left[D_k\right] \xrightarrow[k\to\infty]{} \mathbb{P}_{\mu^{*l}}\left[w_{js}(J) \text{ dominates } J \text{ for all } j \geq 1\right].$$

From this, we also obtain

$$\left\lceil \frac{n}{s} \right\rceil^{-1} \sum_{k=1}^{\lceil n/s \rceil} \mathbb{P}_{\mu^{*l}}\left[D_k\right] \xrightarrow[n\to\infty]{} \mathbb{P}_{\mu^{*l}}\left[w_{js}(J) \text{ dominates } J \text{ for all } j \geq 1\right],$$

which implies the desired inequality. □

Lemma 4.6.4. *Consider the same hypotheses as in Proposition 4.6.2. Then there exists a sparsity $s \in \mathbb{N}_+$ such that for all $l \in \mathbb{N}_+$ and every interval $J \subseteq I$ we have*

$$\mathbb{P}_{\mu^{*l}}\left[w_{js}(J) \text{ dominates } J \text{ for all } j \geq 1\right] \geq 1/24.$$

Proof. Notice that the probability measure ν on S^1, which is the unique μ-stationary probability measure on S^1, is also the unique μ^{*l}-stationary measure on S^1 for each $l \geq 1$. Since ν is non-atomic we have that

$$\nu(\{\xi \in S^1 : 0 < d(\xi, I) < \varepsilon\}) \xrightarrow[\varepsilon\to 0]{} 0.$$

We see from this that there exists $\varepsilon > 0$, that does not depend on l, such that

$$\nu(\{\xi \in S^1 : d(\xi, I) > \varepsilon\}) \geq (1 - \nu(I))/2.$$

Let us define $\Xi = \{\xi \in S^1 : d(\xi, I) > \varepsilon\}$, and recall that we are supposing $\nu(I) < 1/2$. Together with the above, this implies that $\nu(\Xi) > 1/4$.

Let $\lambda > 0$ be such that the conclusion of Corollary 4.5.2 is verified for $\mathbb{P} = \mathbb{P}_{\mu^{*l}}$, and choose a sparsity s such that $e^{-\lambda s/2} < \min\{\varepsilon/8, 1/5\}$. Since

$$\sup_{x \in S^1} \mathbb{E}_{\mu^{*l}} \left[d\left(w_n(x), \xi(\mathbf{w})\right)\right] = \sup_{x \in S^1} \mathbb{E}\left[d(w_{ln}(x), \xi(\mathbf{w}))\right] \leq e^{-\lambda l n} \leq e^{-\lambda n} \tag{4.6.1}$$

for all $n \in \mathbb{N}_+$, we can choose λ and s uniform in l.

Denote by l (resp. r) the left (resp. right) endpoint of the interval J, so that we have $J = [l, r]$. We claim that if the interval $w_{js}(J)$ does not dominate J for some $j \geq 1$, then for each $\xi \in \Xi$ we have $\max\{d(w_{js}(l), \xi), d(w_{js}(r), \xi)\} \geq \varepsilon$. Indeed, if $w_{js}(J)$ does not dominate J, then either $w_{js}(J) \subseteq J$ or $\{w_{js}(l), w_{js}(r)\} \cap J \neq \varnothing$. In both cases we obtain that $\{w_{js}(l), w_{js}(r)\} \cap I \neq \varnothing$, and hence there is an endpoint of the interval $w_{js}(J)$ at distance at least ε from Ξ (see Figure 4.2).

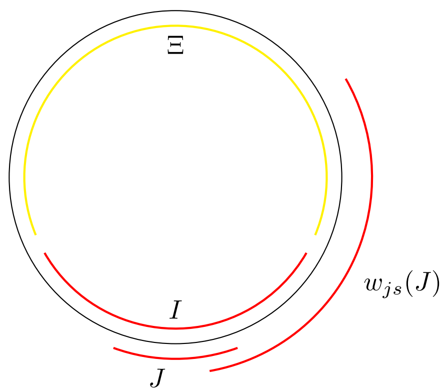

Figure 4.2. The interval $w_{js}(J)$ does not dominate J.

For each $j \geq 1$ denote by A_j the event where $w_{js}(J)$ does not dominate J. By the above paragraph, for every $j \geq 1$ we have

$$\begin{aligned}\nu(\Xi)\mathbb{P}_{\mu^{*l}}\left[A_j \mid \xi \in \Xi\right] &\leq \nu(\Xi)\mathbb{P}_{\mu^{*l}}\left[\max\{d(w_{js}(l), \xi), d(w_{js}(r), \xi)\} > \varepsilon \mid \xi \in \Xi\right] \\ &\leq \mathbb{P}_{\mu^{*l}}\left[\max\{d(w_{js}(l), \xi), d(w_{js}(r), \xi)\} > \varepsilon\right] \leq e^{-\lambda js}\frac{2}{\varepsilon},\end{aligned}$$

where in the last inequality we used Equation (4.6.1) together with the Markov inequality.

Since $e^{-\lambda s/2} < \varepsilon\nu(\Xi)/2$, we see that

$$\mathbb{P}_{\mu^{*l}}\left[A_j \mid \xi \in \Xi\right] \leq e^{-\lambda js}\frac{2}{\varepsilon\nu(\Xi)} \leq e^{-\lambda s/2}e^{-\lambda(j-1)s}.$$

Hence

$$\mathbb{P}_{\mu^{*l}}\left[\bigcup_{j\geq 1} A_j \mid \xi \in \Xi\right] \leq e^{-\lambda s/2}\sum_{j\geq 1} e^{-\lambda(j-1)s} = \frac{e^{-\lambda s/2}}{1-e^{-\lambda s}}$$

which is at most $5/24$ since $e^{-\lambda s/2} \leq 1/5$ and $e^{-\lambda s} \leq 1/25$. Finally, we obtain

$$\begin{aligned}\mathbb{P}_{\mu^{*l}}\left[\bigcup_{j\geq 1} A_j\right] &= \mathbb{P}_{\mu^{*l}}\left[\bigcup_{j\geq 1} A_j \mid \xi(\mathbf{w}) \in \Xi\right] \mathbb{P}_{\mu^{*l}}[\xi(\mathbf{w}) \in \Xi] + \\ &\quad + \mathbb{P}_{\mu^{*l}}\left[\bigcup_{j\geq 1} A_j \mid \xi(\mathbf{w}) \notin \Xi\right] \mathbb{P}_{\mu^{*l}}[\xi(\mathbf{w}) \notin \Xi] \\ &\leq \frac{5}{24} + \nu(S^1 \setminus \Xi) \leq \frac{5}{24} + \frac{3}{4} = \frac{23}{24},\end{aligned}$$

which proves the lemma. □

Proof of Proposition 4.6.2. Lemma 4.6.3 together with Lemma 4.6.4 imply the statement of Proposition 4.6.2 by setting $c = \frac{1}{48s}$. □

4.7 Proof of Theorem F

This section finishes the proof of Theorem F. As in the statement of Theorem F, in this section we fix a countable group G acting proximally, minimally, and topologically non-freely on S^1 and μ a non-degenerate probability measure on G.

By Lemma 4.2.6, we may assume that $\mu(e_G) > 0$. Denote by ν the unique μ-stationary measure on S^1, and denote by $\overline{\nu}$ the unique $\overline{\mu}$ stationary measure where $\overline{\mu}$ is the non-degenerate probability measure on G given by $\overline{\mu}(g) = \mu(g^{-1})$ for each $g \in G$.

Fix an interval $I \subseteq S^1$ such that $\nu(I) < 1/2$. Using Proposition 4.6.2 we can find $0 < c < 1$ and $s \in \mathbb{N}_+$ such that $\mathbb{E}_{\mu^{*l}}[Z^J_{n,s}] \geq cn$ for all $l \in \mathbb{N}_+$, every interval $J \subseteq I$ and all sufficiently large $n \in \mathbb{N}_+$. Since

$$\mathbb{E}_{\mu^{*sl}}[Z^J_{n,1}] = \mathbb{E}_{\mu^{*l}}[Z^J_{ns,s}] \geq cns \geq cn,$$

for all $l \in \mathbb{N}_+$ and sufficiently large $n \in \mathbb{N}_+$, we can replace once and for all μ by μ^{*sl} with some large $l \in \mathbb{N}_+$ (and drop the subscript μ^{*l} from $\mathbb{P}, \mathbb{E}$) so that

- $\mathrm{supp}(\mu)$ contains an element $a \in G \setminus \{e_G\}$ with $\mathrm{supp}(a) \subseteq I$, such that the smallest subinterval $J \subseteq I$ that contains $\mathrm{supp}(a)$ satisfies $\overline{\nu}(J) < c/8$, and
- $\mathbb{E}[Z^J_{n,1}] \geq cn$ for all large $n \in \mathbb{N}_+$.

Notice that the stationary measures ν, $\overline{\nu}$ on S^1 do not change after doing this replacement.

Lemma 4.7.1. *For every $n \in \mathbb{N}_+$, denote by $W_n \in \mathbb{N}_+$ the random variable that counts the number of times $1 \leq k \leq n$ such that*

- *the interval $w_k(J)$ dominates $w_j(J)$ for all $0 \leq j \leq k-1$,*

- $w_k^{-1}(\xi(\mathbf{w}))$ *does not belong to* J, *and*
- *the increment* g_{k+1} *is in* $\{a, e_G\}$.

Then there exists $0 < c' < 1$ *such that* $\mathbb{E}[W_n] \geq c'n$ *for sufficiently large* $n \in \mathbb{N}_+$.

Proof. For every $n \in \mathbb{N}_+$, denote by $\widetilde{W}_n \in \mathbb{N}$ the random variable that counts the number of times $1 \leq k \leq n$ such that

- the interval $w_k(J)$ dominates $w_j(J)$ for all $0 \leq j \leq k-1$, and
- $w_k^{-1}(\xi(\mathbf{w}))$ does not belong to J.

By Proposition 4.5.3, we see that

$$\frac{\mathbb{E}\left[\left|\left\{1 \leq k \leq n : w_k^{-1}(\xi(\mathbf{w})) \notin J\right\}\right|\right]}{n} \geq 1 - 2\overline{\nu}(J) \geq 1 - \frac{c}{4}, \tag{4.7.1}$$

for all sufficiently large $n \in \mathbb{N}_+$. The bound $\mathbb{E}[Z_{n,1}^J] \geq cn$ and Equation (4.7.1) imply that there exists $c'' \in (0,1)$ such that $\mathbb{E}[\widetilde{W}_n] \geq c''n$ for all sufficiently large $n \in \mathbb{N}_+$.

Notice that

$$\widetilde{W}_n = \sum_{k=1}^{n} 1_{\widetilde{C}_k} \quad \text{and} \quad W_n = \sum_{k=1}^{n} 1_{C_k},$$

where

$$\widetilde{C}_k = \{w_k(J) \text{ dominates } w_j(J) \text{ for } 0 \leq j \leq k-1, \text{ and } w_k^{-1}(\xi(\mathbf{w})) \notin J\}$$

and

$$C_k = \widetilde{C}_k \cap \{g_{k+1} \in \{a, e_G\}\}$$

for every $1 \leq k \leq n$. Since the event $\{\mathbf{w} \in G^{\mathbb{N}} : g_{k+1} \in \{a, e_G\}\}$ is independent from $\widetilde{C}_k$ under $\mathbb{P}$ we deduce that

$$\mathbb{E}[W_n] = (\mu(a) + \mu(e_G))\mathbb{E}[\widetilde{W}_n] \geq (\mu(a) + \mu(e_G))c''n$$

for all sufficiently large $n \in \mathbb{N}_+$, showing the conclusion for the constant $c' = (\mu(a)+\mu(e_G))c''$. □

We recall the following basic fact about random variables, that we will use below.

Lemma 4.7.2. *Let* $0 \leq X \leq 1$ *be a real-valued random variable with mean* $\mathbb{E}[X] > \lambda > 0$. *Then*

$$\mathbb{P}[X > \lambda/2] \geq \lambda/2.$$

Proof. The statement follows from the inequality

$$\lambda < \mathbb{E}[X] \leq \lambda/2\mathbb{P}[X \leq \lambda/2] + \mathbb{P}[X > \lambda/2] \leq \lambda/2 + \mathbb{P}[X > \lambda/2].$$ □

Finally, we present the proof of Theorem F.

Proof of Theorem F. Just as in the statement of Lemma 4.7.1, for every $n \geq 1$ denote by $W_n \in \mathbb{N}_+$ the random variable that counts the number of times $1 \leq k \leq n$ such that

- the interval $w_k(J)$ dominates $w_j(J)$ for all $0 \leq j \leq k-1$,
- $w_k^{-1}(\xi(\mathbf{w}))$ does not belong to J, and
- the increment g_{k+1} is in $\{a, e_G\}$.

For ν-almost every $\xi \in S^1$ and $n \in \mathbb{N}_+$ we apply Lemma 4.7.2 to the random variable

$$\mathbf{w} \in (G^{\mathbb{N}}, \mathbb{P}^{\xi}) \mapsto \frac{W_n}{n} \in [0,1],$$

and deduce that

$$\mathbb{P}^{\xi}\left[\frac{W_n}{n} > \frac{\mathbb{E}^{\xi}[W_n]}{2n}\right] \geq \frac{\mathbb{E}^{\xi}[W_n]}{2n}.$$

Now consider the random variable

$$\xi \in (S^1, \nu) \mapsto \frac{\mathbb{E}^{\xi}[W_n]}{n} \in [0,1]$$

and apply Lemmas 4.7.1 and 4.7.2 to see that $\nu(\Xi_n) > c'/2$, where

$$\Xi_n = \left\{\xi \in S^1 : \frac{\mathbb{E}^{\xi}[W_n]}{n} \geq \frac{c'}{2}\right\}.$$

From this, we conclude that

$$\mathbb{P}^{\xi}\left[\frac{W_n}{n} > \frac{c'}{2}\right] \geq \frac{c'}{2} \tag{4.7.2}$$

for ν-almost every $\xi \in \Xi_n$.

For every $n \in \mathbb{N}_+$ and ν-almost every $\xi \in \Xi_n$, consider the set of trajectories $\mathbf{w} = (w_n)_{n\geq 0} \in G^{\mathbb{N}}$ such that $W_n(\mathbf{w})/n > c'/2$. To each such sample path $\mathbf{w}$ we associate a maximal set of indices $1 \leq i_1 < \cdots < i_k \leq n$ of size $k = k(\mathbf{w})$ such that for every $1 \leq r \leq k$ we have that

- the interval $w_{i_r - 1}(J)$ dominates $w_l(J)$ for all $0 \leq l < i_r - 1$, and
- $g_{i_r} \in \{a, e_G\}$ and $w_{i_r}^{-1}(\xi) \notin J$.

Define $\mathbf{x}$ as the $(n-k)$-tuple consisting of all increments of $\mathbf{w}$ at times in $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. By definition, the collection $q(\mathbf{w}) = (\mathbf{x}, i_1, \ldots, i_k)$ is ξ-good, and we have $k(\mathbf{w}) \geq W_n > c'n/2$.

Denote by $A_{n,\xi}$ the set of collections obtained in this way. We claim that the collections $A_{n,\xi}$ satisfy the conditions of Theorem 4.3.2. Indeed, since $A_{n,\xi}$ is composed of ξ-good collections, thanks to Proposition 4.4.5 we have that Conditions (ii), (iii) and (v) are satisfied. Moreover, for every $q = (\mathbf{x}, i_1, \ldots, i_k) \in A_{n,\xi}$ we have $k \geq c'n/2$ by the previous paragraph. Therefore, Condition (i) is also satisfied. Finally, from Inequality (4.7.2) we get that

$$\mathbb{P}^{\xi}\left[\bigcup_{q \in A_{n,\xi}} \bigcup_{\mathbf{y} \in T^{a,e_G}(q)} [\mathbf{y}]\right] \geq c'/2,$$

so Condition (iv) also holds. The hypotheses of Theorem 4.3.2 are satisfied, and hence we have finished the proof of Theorem F. □

4.8 Proof of Theorem G

This section is devoted to the proof of Theorem G. Subsection 4.8.1 adapts V. Kaimanovich's construction of a μ-boundary on Thompson's group F to our context, and Subsection 4.8.2 uses it to prove Theorem G. As in the statement of Theorem G, in this section G is a countable subgroup of $\mathrm{PAff}_0(S^1)$ acting minimally, proximally and topologically non-freely on S^1, and μ is a non-degenerate probability measure on G such that the sum $\sum_{g\in G}\mu(g)|\mathbf{Br}_g|$ is finite, where $\mathbf{Br}_g$ is the finite set of breakpoints of $g \in G$.

Recall that the group $\mathrm{PAff}_0(S^1)$ of *piecewise affine* orientation-preserving homeomorphisms of $S^1 \cong \mathbb{R}/\mathbb{Z}$ is the group of all $g \in \mathrm{Homeo}_0(S^1)$ such that there exists a finite subset $D \subseteq S^1$ such that g restricted to every connected component C of $S^1 \setminus D$ is of the form $g(x) = ax + b$ for some $a > 0$ and $b \in \mathbb{R}/\mathbb{Z}$. Thus, for every $g \in \mathrm{PAff}_0(S^1)$ the derivative g' is defined outside a finite set and is locally constant. The points of S^1 where the derivative of g is not defined are called the *breakpoints* of g. We denote by $\mathbf{Br} \subseteq S^1$ the countable set of breakpoints of elements in G.

4.8.1 The breakpoint boundary

Given an element $g \in G$, define a finitely supported function $\mathcal{C}_g\colon \mathbf{Br} \to \mathbb{R}$ by setting

$$\mathcal{C}_g(x) = \log\left((g^{-1})'(x^+)\right) - \log\left((g^{-1})'(x^-)\right)$$

for $x \in \mathbf{Br}$, where $(g^{-1})'(x^+)$ (resp. $(g^{-1})'(x^-)$ is the left (resp. right) derivative of g^{-1} at x. That is, $\mathcal{C}_g(x)$ is the derivative jump of g^{-1} at x. This definition differs slightly from those used in [Kai17, Sta21], but this difference is necessary since we consider *right* random walks and the *left* action of G on S^1.

Denote the set of all (not necessarily finitely supported) functions $\mathbf{Br} \to \mathbb{R}$ by $\mathbb{R}^{\mathbf{Br}}$. Define a left action of G on $\mathbb{R}^{\mathbf{Br}}$ by

$$(g, \mathcal{C}) \mapsto \left(S_g\mathcal{C}\colon x \in \mathbf{Br} \mapsto \mathcal{C}(g^{-1}(x))\right).$$

By the chain rule, we have

$$\mathcal{C}_{gh}(x) = \log\left((h^{-1})'(g^{-1}(x)^+)\right) - \log\left((h^{-1})'(g^{-1}(x)^-)\right) + \log\left((g^{-1})'(x^+)\right) - \log\left((g^{-1})'(x^-)\right)$$

for all $g, h \in G$ and $x \in \mathbf{Br}$, so that

$$\mathcal{C}_{gh} = \mathcal{C}_g + S_g\mathcal{C}_h. \tag{4.8.1}$$

Let us define a second left action of G on $\mathbb{R}^{\mathbf{Br}}$ by $(g, \mathcal{C}) \mapsto \mathcal{C}_g + S_g\mathcal{C}$, so we have $\mathcal{C}_{gh} = g.\mathcal{C}_h$ for all $g, h \in G$. This is the action on $\mathbb{R}^{\mathbf{Br}}$ that will define a non-trivial boundary for G.

To prove the transience of the random walks on G-orbits of elements of $\mathbf{Br}$ we emulate [Kai17, Theorem 25], for which we need a comparison lemma for Markov operators due to [BLP77]; see also the proposition at the end of Section 4 in [Var83] for a more general version of this result.

Proposition 4.8.1. *Let $P_1(\cdot,\cdot), P_2(\cdot,\cdot)$ be doubly stochastic kernels on a countable set X such that $P_2(\cdot,\cdot)$ is symmetric and there exists $\varepsilon > 0$ such that*

$$P_1(x,y) \geq \varepsilon P_2(x,y) \text{ for all } x,y \in X.$$

Then the Markov process determined by P_1 and started at $x \in X$ is transient if the Markov process determined by P_2 and started at $x \in X$ is transient.

Lemma 4.8.2. *For every $x \in S^1$ the μ-random walk on $\mathrm{Orb}_G(x)$ started at x is transient.*

Proof. Fix $x \in S^1$. Since G acts proximally and minimally, using Proposition 4.2.10 we can find $f, g \in G$ such that there are disjoint intervals $I_1, I_2, J_1, J_2 \subseteq S^1$ with $x \notin I_1 \cup I_2 \cup J_1 \cup J_2$ and

$$f(S^1 \setminus I_2) \subseteq I_1, \quad g(S^1 \setminus J_2) \subseteq J_1.$$

By Klein's ping-pong lemma, f and g generate a free subgroup of G and $\langle f, g\rangle$ acts freely on $\mathrm{Orb}_{\langle f,g\rangle}(x)$. Let $\tilde{\mu}$ be the uniform measure on $\{f, f^{-1}, g, g^{-1}\}$. The $\tilde{\mu}$-random walk on $\mathrm{Orb}_{\langle f,g\rangle}(x)$ starting at x is transient since it corresponds to a simple random walk on an infinite tree of valence 4. Let $n \in \mathbb{N}$ be such that $f, g \in \mathrm{supp}(\mu^{*n})$. Then there exists $\varepsilon > 0$ such that $\mu^{*n} \geq \varepsilon\tilde{\mu}$ elementwise, and we obtain from Proposition 4.8.1 that the μ^{*n}-random walk on $\mathrm{Orb}_G(x)$ started at x is transient. We conclude that the μ-random walk on $\mathrm{Orb}_G(x)$ starting at x is transient too. □

The following lemma is reminiscent of the stabilization of lamp configurations for random walks on wreath products; see the references in Subsection 4.1.1 and also [Sta21, Lemma 7.2]. This is the only point in the construction of the breakpoint boundary where the moment condition on μ is used. For $\mathcal{C} \in \mathbb{R}^{\mathbf{Br}}$ we denote $\mathrm{supp}(\mathcal{C}) = \{x \in \mathbf{Br} : \mathcal{C}(x) \neq 0\}$.

Lemma 4.8.3. *Suppose that $\sum_{g\in G} \mu(g)|\mathbf{Br}_g| < \infty$. Then $\mathbb{P}$-almost surely and every $x \in \mathbf{Br}$ there is $N \geq 1$ such that $\mathcal{C}_{w_n}(x) = \mathcal{C}_{w_{n+1}}(x)$ for each $n \geq N$. Hence, the configurations $(\mathcal{C}_{w_n})_{n\geq 0}$ converge pointwise to a map $\mathcal{C}_\infty(\mathbf{w}) \in \mathbb{R}^{\mathbf{Br}}$.*

Proof. The equation

$$\mathcal{C}_{w_{n+1}} = \mathcal{C}_{w_n} + S_{w_n}\mathcal{C}_{g_{n+1}}$$

implies that, for every $x \in \mathbf{Br}$, $\mathcal{C}_{w_{n+1}}(x) = \mathcal{C}_{w_n}(x)$ if and only if $w_n^{-1}(x) \notin \mathrm{supp}(\mathcal{C}_{g_{n+1}})$. Thus, the configurations $\mathcal{C}_{w_n}$ stabilize as $n \to \infty$ to some $\mathcal{C}_\infty(\mathbf{w}) \in \mathbb{R}^{\mathbf{Br}}$ if for every $x \in \mathbf{Br}$, we have $w_n^{-1}(x) \in \mathrm{supp}(\mathcal{C}_{g_{n+1}})$ for only finitely many $n \in \mathbb{N}$. By the Borel-Cantelli lemma, this holds whenever

$$\begin{aligned}\sum_{n\geq 0} \mathbb{P}\left[w_n^{-1}(x) \in \mathrm{supp}(\mathcal{C}_{g_{n+1}})\right] &= \sum_{n\geq 0}\sum_{g\in G} \mu(g)\mathbb{P}\left[w_n^{-1}(x) \in \mathrm{supp}(\mathcal{C}_g)\right] \\ &= \sum_{n\geq 0}\sum_{g\in G}\sum_{y\in\mathrm{supp}(\mathcal{C}_g)} \mu(g)\mathbb{P}\left[w_n^{-1}(x) = y\right]\end{aligned}$$

is finite for all $x \in \mathbf{Br}$.

The μ-random walk $(w_n)_{n\geq 0}$ on G induces a Markov chain on $\mathbf{Br}$ with transition probabilities $p(x,y) = \mathbb{P}\left[w_1^{-1}(x) = y\right]$. Denote by $\overline{p}$ its reflected kernel, defined by $\overline{p}(x,y) = p(y,x)$ for all $x, y \in \mathbf{Br}$. Notice that for $x, y \in \mathbf{Br}$, the quantity $\sum_{n\geq 0} \overline{p}^{*n}(y,x)$ is the expected number of visits to x of the Markov chain defined by $\overline{p}$ and starting at y. Hence

$$\sum_{n\geq 0} \overline{p}^{*n}(y,x) = q(y,x) \sum_{n\geq 0} \overline{p}^{*n}(x,x)$$

where $q(y,x)$ is the probability that the Markov chain defined by $\overline{p}$ and started at y hits x. In particular,

$$\sum_{n\geq 0} \overline{p}^{*n}(y,x) \leq \sum_{n\geq 0} \overline{p}^{*n}(x,x),$$

so an upper bound for $\sum_{n\geq 0} \mathbb{P}\left[w_n^{-1}(x) \in \operatorname{supp}(\mathcal{C}_{g_{n+1}})\right]$ is given by

$$\begin{aligned}\sum_{g\in G} \sum_{y\in \operatorname{supp}(\mathcal{C}_g)} \mu(g) \left(\sum_{n\geq 0} \overline{p}^{*n}(x,x)\right) &= \left(\sum_{n\geq 0} \overline{p}^{*n}(x,x)\right) \left(\sum_{g\in G} \mu(g) \left|\mathbf{Br}_{g^{-1}}\right|\right) \\ &= \left(\sum_{n\geq 0} \overline{p}^{*n}(x,x)\right) \left(\sum_{g\in G} \mu(g) \left|\mathbf{Br}_{g}\right|\right)\end{aligned}$$

By Lemma 4.8.2, the sum $\sum_{n\geq 0} \overline{p}^{*n}(x,x)$ is finite for any $x \in \mathbf{Br}$, and by hypothesis the term $\sum_{g\in G} \mu(g)|\mathbf{Br}_g|$ is also finite. This proves the lemma. $\square$

Denote by $\widetilde{\nu}$ the pushforward measure of $\mathbb{P}$ through $\mathcal{C}_\infty \colon G^{\mathbb{N}} \to \mathbb{R}^{\mathbf{Br}}$. Then the space $(\mathbb{R}^{\mathbf{Br}}, \widetilde{\nu})$ is a μ-boundary, that we call the *breakpoint boundary*. One can prove the non-triviality of this boundary by following steps similar to those in the proof of [Sta21, Lemma 7.3], or, alternatively, this will follow from the proof of Theorem G in the next subsection.

4.8.2 Bounded harmonic functions not coming from (S^1, ν)

Recall that we denote by (S^1, ν) the μ-boundary coming from the natural action of G on the circle. In this subsection, we provide a family of harmonic functions defined through the breakpoint boundary $(\mathbb{R}^{\mathbf{Br}}, \tilde{\nu})$ that cannot be obtained from (S^1, ν) via the Poisson transform.

Lemma 4.8.4. *Let G be a countable subgroup of $\mathrm{PAff}_0(S^1)$ whose action on S^1 is minimal, proximal and topologically non-free, and let μ be a non-degenerate probability measure on G such that $\sum_{g\in G} \mu(g)|\mathbf{Br}_g| < \infty$.*

Then for every $n \in \mathbb{N}_+$ there exists a μ-harmonic function $\zeta_n \colon G \to [0,1]$ with $\zeta_n(e_G) > 1/2$ and an element $a_n \in G$ such that

$$\operatorname{diam}(\operatorname{supp}(a_n)) \xrightarrow[n\to\infty]{} 0 \quad \textit{and} \quad \zeta_n(a_n) \xrightarrow[n\to\infty]{} 0.$$

Proof. Consider an element $a \in G \setminus \{e_G\}$ such that $\operatorname{supp}(a)$ is strictly contained in S^1 (such an element exists because the action of G is topologically non-free). Fix $y \in \partial(\operatorname{supp}(a))$, so that $y \in \mathbf{Br}$. Since the action of G on S^1 is minimal and proximal, there exists a sequence $(t_n)_{n\geq 1} \subseteq G$

such that $\operatorname{diam}(t_n(\operatorname{supp}(a))) \leq 1/n$ for all $n \in \mathbb{N}_+$. Since the measure $\tilde{\nu}$ is non-zero, for every $n \in \mathbb{N}_+$ there exists a bounded open set $U_n \subseteq \mathbb{R}$ such that the bounded function $\zeta_n : G \to [0,1]$ defined on $g \in G$ by

$$\zeta_n(g) = \mathbb{P}_g\left[\left\{\mathbf{w} \in G^{\mathbb{N}} : \mathcal{C}_\infty(\mathbf{w})(t_n(y)) \in U_n\right\}\right]$$

satisfies $\zeta_n(e_G) > 1/2$. Notice that the event $\{\mathbf{w} \in G^{\mathbb{N}} : \mathcal{C}_\infty(\mathbf{w})(t_n(y)) \in U_n\}$ is shift-invariant up to $\mathbb{P}$-measure zero, and hence the function ζ_n is μ-harmonic.

For each $n \in \mathbb{N}_+$ define $b_n = t_n a t_n^{-1}$. Then we have

$$\zeta_n(b_n^j) = \mathbb{P}\left[\mathcal{C}_\infty(b_n^j \mathbf{w})(t_n(y)) \in U_n\right]$$

for every $j \in \mathbb{N}_+$. Using Equation (4.8.1) we get

$$\mathcal{C}_\infty(b_n^j \mathbf{w})(t_n(y)) = \mathcal{C}_{b_n^j}(t_n(y)) + S_{b_n^j}\mathcal{C}_\infty(\mathbf{w})(t_n(y)) = \mathcal{C}_{b_n^j}(t_n(y)) + \mathcal{C}_\infty(\mathbf{w})(t_n(y)), \tag{4.8.2}$$

where in the last equality we used that b_n^j fixes $t_n(y)$. By iterating Equation (4.8.1) and using that a^{-j} fixes y, we see that

$$\begin{aligned}
\mathcal{C}_{b_n^j}(t_n(y)) = \mathcal{C}_{t_n a^j t_n^{-1}}(t_n(y)) &= \mathcal{C}_{t_n a^j}(t_n(y)) + S_{t_n a^j}\mathcal{C}_{t_n^{-1}}(t_n(y)) \\
&= \mathcal{C}_{t_n a^j}(t_n(y)) + \mathcal{C}_{t_n^{-1}}(a^{-j}(y)) \\
&= \mathcal{C}_{t_n a^j}(t_n(y)) + \mathcal{C}_{t_n^{-1}}(y) \\
&= \mathcal{C}_{t_n}(t_n(y)) + S_{t_n}\mathcal{C}_{a^j}(t_n(y)) + \mathcal{C}_{t_n^{-1}}(y) \\
&= \mathcal{C}_{t_n}(t_n(y)) + \mathcal{C}_{a^j}(y) + \mathcal{C}_{t_n^{-1}}(y) = j\mathcal{C}_a(y).
\end{aligned}$$

The above together with Equation (4.8.2) shows that $\zeta_n(b_n^j) = \mathbb{P}\left[j\mathcal{C}_a(y) + \mathcal{C}_\infty(\mathbf{w})(t_n(y)) \in U_n\right]$. Next, note that $\mathcal{C}_a(y) \neq 0$ by the choice of y. Since U_n is bounded, there exists $j_n \in \mathbb{N}_+$ sufficiently large with $\zeta_n(b_n^{j_n}) < 1/n$. Denote $a_n = b_n^{j_n}$, so the diameter of

$$\operatorname{supp}(a_n) = \operatorname{supp}(b_n) = t_n(\operatorname{supp}(a))$$

goes to 0 as $n \to \infty$ and also $\zeta_n(a_n) \xrightarrow[n\to\infty]{} 0$. □

We can now present the proof of Theorem G.

Proof of Theorem G. Towards a contradiction, let us suppose that the breakpoint boundary is a G-equivariant quotient of (S^1, ν). For every $n \in \mathbb{N}_+$ consider μ-harmonic functions $\zeta_n : G \to [0,1]$ and elements $a_n \in G$ as in Lemma 4.8.4. Then there exist functions $F_n \in L^\infty(S^1, \nu)$ such that

$$\zeta_n(g) = \int_{S^1} F_n(g(x))\,\mathrm{d}\nu(x)$$

for all $g \in G$. Since the Poisson transform is an isometry (Theorem 4.2.2), we have $\|F_n\|_\infty \leq 1$.

Write $I_n = \operatorname{supp}(a_n)$, so $F_n(a_n(x)) = F_n(x)$ whenever $x \notin I_n$ and

$$\begin{aligned}
|\zeta_n(a_n) - \zeta_n(e_G)| &\leq \int_{S^1 \setminus I_n} |F_n(a_n(x)) - F_n(x)|\,\mathrm{d}\nu(x) + \int_{I_n} |F_n(a_n(x)) - F_n(x)|\,\mathrm{d}\nu(x) \\
&= \int_{I_n} |F_n(a_n(x)) - F_n(x)|\,\mathrm{d}\nu(x) \leq 2\nu(I_n)\|F_n\|_\infty \leq 2\nu(I_n) \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{4.8.3}$$

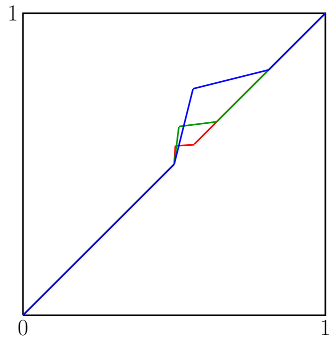

Figure 4.3. The maps a_n for $y = 1/2$ and $n = 2$ (blue), $n = 3$ (green) and $n = 4$ (red).

However, from Lemma 4.8.4 we have that $\liminf_{n\to\infty}|\zeta_n(a_n) - \zeta_n(e_G)| \geq 1/2$. This contradicts the convergence in (4.8.3). □

Remark. When G is Thompson's group T we can provide a single μ-harmonic function that does not arise from (S^1, ν). Indeed, in this case $\mathbf{Br} = \mathbb{Z}[1/2]/\mathbb{Z}$ and, after defining configurations using logarithms in base 2, we have a μ-boundary $(\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}, \widetilde{\nu})$ where $\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}$ is the space of functions from $\mathbb{Z}[1/2]/\mathbb{Z}$ to $\mathbb{Z}$. Pick any $y \in \mathbb{Z}[1/2]/\mathbb{Z}$ and a $k \in \mathbb{Z}$ so that the function defined by $\zeta(g) = \mathbb{P}_g\left[\mathcal{C}_\infty(\mathbf{w})(y) = k\right]$ for each $g \in G$ satisfies $\zeta(e_G) > 0$.

Consider for every $n \geq 1$ the element $a_n \in T$ defined by

$$a_n(x) = \begin{cases} y + 2^n(x-y) & \text{if } y \leq x < y + 2^{-2n} \\ y + 2^{-n} - 2^{-3n} + 2^{-n}(x-y) & \text{if } y + 2^{-2n} < x < y + 2^{-2n} + 2^{-n} \\ x & \text{elsewhere,} \end{cases}$$

see Figure 4.3.

Then, we have that

- $a_n(y) = y$,
- $\operatorname{supp}(a_n)$ is a dyadic interval containing y and of length $2^{-n} + 2^{-2n}$, and
- the derivative jump of a_n at y is equal to 2^n.

This implies that both $\operatorname{diam}(\operatorname{supp}(a_n))$ and $\zeta(a_n)$ converge to 0 as n goes to infinity. From this point, one can continue just as in the proof of Theorem G to conclude that there is no $F \in L^\infty(S^1, \nu)$ such that $\zeta(g) = \int_{S^1} F(g(x))\, \mathrm{d}\nu(x)$ for all $g \in G$.

Chapter 5

Stationary boundaries on the space of amenable subgroups

This chapter corresponds to the preprint [CGVS26], **and represents joint work with Anna Cascioli (Universität Münster) and Eduardo Silva (Universität Münster)**.

We give a sufficient condition for a countable group G to possess a probability measure μ that admits a non-trivial μ-boundary modeled in the space $\mathrm{Sub}_{\mathrm{am}}(G)$ of amenable subgroups of G. In particular, for such μ the space $\mathrm{Sub}_{\mathrm{am}}(G)$ is not uniquely μ-stationary. This contrasts with a theorem of Hartman-Kalantar, which states that a countable group G is C*-simple if and only if there exists $\mu \in \mathrm{Prob}(G)$ such that $\mathrm{Sub}_{\mathrm{am}}(G)$ is uniquely μ-stationary [HK23]. Our criterion applies to (permutational) wreath products, which include groups that are C*-simple, and to Thompson's group F, whose C*-simplicity is equivalent to its non-amenability and therefore remains an open problem. We also show that any non-trivial μ-boundary modeled on $\mathrm{Sub}_{\mathrm{am}}(G)$ is supported on amenable normalish subgroups, in the sense of Breuillard-Kalantar-Kennedy-Ozawa [BKKO17]. As a consequence, we conclude that a countable group with no finite normal subgroups and no amenable normalish subgroups acts essentially freely on all its Poisson boundaries.

5.1 Introduction

The space $\mathrm{Sub}(G)$ of subgroups of a discrete countable group G is naturally identified with a closed subset of $\{0,1\}^G$ endowed with the product topology. The induced topology, known as the *Chabauty topology*, makes $\mathrm{Sub}(G)$ a compact metrizable space, and the conjugation action of G on its subgroups induces an action by homeomorphisms $G \curvearrowright \mathrm{Sub}(G)$. A central object in the study of this dynamical system is the class of G-invariant probability measures on $\mathrm{Sub}(G)$, known as *invariant random subgroups* (IRSs). Notable applications of IRSs include rigidity results such as the Stuck–Zimmer theorem [SZ94], approximation results for the ℓ^2-Betti numbers of semisimple Lie groups [ABB+17], and the realization of entropy spectra of stationary actions for certain classes of countable groups [Bow14, HY18].

The set $\mathrm{Sub}_{\mathrm{am}}(G)$ of amenable subgroups of G is closed in $\mathrm{Sub}(G)$, and a reason to study the dynamics of $G \curvearrowright \mathrm{Sub}_{\mathrm{am}}(G)$ is its connection with the C*-simplicity of G. A countable group G is said to be *C*-simple* if its reduced C*-algebra $C_r^*(G)$ contains no non-trivial closed ideals, where the *reduced C*-algebra* $C_r^*(G)$ *of* G is the norm closure of the linear span of the unitary operators defined by the left regular representation of G on $\ell^2(G)$. This property has been well studied from the perspective of operator algebras and group theory [Pow75, BCdlH94]. A result of M. Kennedy [Ken20], following his work with Breuillard-Kalantar-Ozawa [KK17, BKKO17], states that a countable group G is C*-simple if and only if the only minimal closed invariant subset of $\mathrm{Sub}_{\mathrm{am}}(G)$ is the one composed of the trivial subgroup $\{e_G\}$.

The statement of a measurable version of the previous criterion involves stationary probability measures rather than invariant ones. Recall that a probability measure $\mu \in \mathrm{Prob}(G)$ is *non-degenerate* if the semigroup generated by its support is all of G, and that a measurable action of G on a probability space (Z, η) is *μ-stationary* if $\eta = \sum_{g\in G} \mu(g) g_*\eta$. A μ-stationary probability measure on $\mathrm{Sub}_{\mathrm{am}}(G)$ is called an *amenable μ-stationary random subgroup* (or *amenable μ-SRS).* This notion turns out to be useful to characterize C*-simplicity of countable groups by work of Hartman-Kalantar [HK23], who proved that a countable group G is C*-simple if and only if there exists a non-degenerate probability measure $\mu \in \mathrm{Prob}(G)$ such that $\mathrm{Sub}_{\mathrm{am}}(G)$ admits a unique μ-stationary probability measure (which must be $\delta_{\{e_G\}}$). In other words, C*-simple groups possess distinguished probability measures that witness their C*-simplicity. In this chapter we address the existence of probability measures with the opposite property.

Question 5.1.1. Let G be a countable group. Does there exist a non-degenerate probability measure μ on G that $\mathrm{Sub}_{\mathrm{am}}(G)$ admits a μ-SRS distinct from $\delta_{\{e_G\}}$?

As observed in [HK23], if G is not C*-simple then every non-degenerate probability measure μ on G has the above property. It follows that Question 5.1.1 is of interest only when G is C^*-simple. Moreover, Hartman-Kalantar show that there are classes of C^*-simple groups for which no probability measure μ admits an amenable μ-SRS distinct from $\delta_{\{e_G\}}$. This holds for hyperbolic groups, mapping class groups and linear groups [HK23, Theorems 4.12, 6.5 & 6.7, Example 6.6], assuming that their amenable radical (that is, their maximal normal amenable subgroup) is trivial. This discussion suggests a graded notion of C*-simplicity, where groups admitting no non-trivial amenable μ-SRSs can be viewed as "more" C*-simple, in the sense that their stationary dynamics on $\mathrm{Sub}_{\mathrm{am}}(G)$ are highly constrained.

Our results deal with the following class of μ-SRSs.

Definition 5.1.2. *Let μ be a probability measure on a countable group G and denote by $\mathbb{P}_\mu$ the law of the right μ-random walk on G. A μ-stationary probability measure η on the space* $\mathrm{Sub}_{\mathrm{am}}(G)$ *of amenable subgroups of G is called an* amenable boundary μ-stationary random subgroup *if, for $\mathbb{P}_\mu$-almost every trajectory $(w_n)_{n\geq 0} \in G^{\mathbb{N}}$, the sequence $(w_{n*}\eta)_{n\geq 0}$ converges in the weak*-topology to a Dirac mass in* $\mathrm{Sub}_{\mathrm{am}}(G)$.

Equivalently, a probability measure η is an amenable boundary μ-SRS if $(\mathrm{Sub}_{\mathrm{am}}(G), \eta)$ is a μ-boundary, in the sense that it is a G-equivariant quotient of the Poisson boundary of (G, μ).

Our first result provides a condition that guarantees the existence of μ-boundaries on $\mathrm{Sub}(G)$ for *some* non-degenerate measure $\mu \in \mathrm{Prob}(G)$.

Theorem H. *Let G be a countable group. Suppose that there exists a non-trivial subgroup H such that for all finite subsets $Q, Z \subseteq G$ there is $b \in G$ such that both bZ and $b^{-1}Z$ are contained in*

$$\left\{g \in G : Q \cap H = Q \cap gHg^{-1}\right\}.$$

Then there exists a non-degenerate, symmetric and finite-entropy probability measure μ on G such that $\overline{\mathrm{Orb}_G(H)}$ supports a μ-boundary SRS distinct from $\delta_{\{e_G\}}$.

In particular, if H is amenable then G admits an amenable boundary μ-SRS distinct from $\delta_{\{e_G\}}$.

The condition appearing in the statement can be viewed as a strong form of recurrence for the action of G on the orbit of H (see Remark 5.3.2). The strategy we use to construct μ is the method of record times used by Frisch-Hartman-Tamuz-Vahidi Ferdowsi [FHTVF19] to prove that any non-hyper-FC-central group admits non-degenerate probability measures with a non-trivial Poisson boundary.

In Proposition 5.3.2, we also present a shorter and more direct argument for wreath products, which differs from the general proof of Theorem H. Recall that if A is a C^*-simple group, then for every countable group B the wreath product $A \wr B = \bigoplus_B A \rtimes B$ is C^*-simple. Proposition 5.3.2 provides examples of C^*-simple groups for which any probability measure witnessing C*-simplicity, as in the main theorems of [HK23] must necessarily have an infinite support (see Corollary 5.3.3). To the best of our knowledge, these are the first examples of this kind. A. Erschler and J. Frisch communicated to us that they independently obtained Proposition 5.3.2.

The criterion in Theorem H also applies to Thompson's group F, the group of piecewise dyadically affine homeomorphisms of the interval.

Corollary I. *There exists a non-degenerate, symmetric and finite-entropy probability measure μ on Thompson's group F such that $\mathrm{Sub}_{\mathrm{ab}}(F) = \{H \in \mathrm{Sub}(G) : H \text{ is abelian}\}$ supports a non-atomic μ-boundary.*

Remarks.

- The minimal closed subsystems of $\mathrm{Sub}(F)$ and the IRSs of F are supported on normal subgroups (see [LBMB18, Theorem 1.7, (i)] and [DM14] respectively), which are the subgroups of F containing the (non-abelian) derived subgroup $[F, F]$. Corollary I shows that the dynamics of $\mathrm{Sub}_{\mathrm{am}}(F)$ are nevertheless rich enough to support non-trivial μ-SRSs. A well-known open question in group theory asks whether F is amenable [CFP96], which is equivalent to F not being C*-simple by [LBMB18, Theorem 1.6]. If F where non-amenable, it would follow that F belongs to a class of C*-simple groups whose amenable subgroup dynamics are richer than those of hyperbolic groups or mapping class groups.

- The proof of Corollary I applies word by word to the finitely presented non-amenable group of piecewise projective homeomorphisms of $\mathbb{R}$ constructed by Lodha-Moore [LM16], which is known to be C*-simple [LBMB18, Theorem 1.10].

A subgroup H of G is said to be *normalish* if $\bigcap_{z\in Z} zHz^{-1}$ is infinite for every finite subset $Z \subseteq G$. This notion was introduced in [BKKO17, Section 6], where the absence of normalish subgroups is shown to imply C*-simplicity. Our second main result connects it to Question 5.1.1.

Theorem J. *Let μ be a non-degenerate probability measure on a countable group G and let η be a μ-stationary probability measure on $\mathrm{Sub}(G)$. Suppose that $(\mathrm{Sub}(G), \eta)$ is a μ-boundary of G, and that η is not a Dirac mass on a finite normal subgroup of G. Then η is supported on normalish subgroups of G.*

As a consequence we obtain the next result.

Corollary K. *Let G be a countable group with no finite normal subgroups and no amenable normalish subgroups. Then for every non-degenerate $\mu \in \mathrm{Prob}(G)$, the action of G on the Poisson boundary of (G, μ) is essentially free.*

Proof. Denote by $(\partial_\mu G, \nu_\mu)$ the Poisson boundary of (G, μ). R. Zimmer showed that the action of a countable group G on its Poisson boundary is amenable [Zim78, Corollary 5.3], and hence the point stabilizers $\mathrm{Stab}_G(z)$, $z \in \partial_\mu G$ are ν_μ-almost surely amenable [ADR00, Corollary 5.3.33]. The pushforward of ν_μ under the stabilizer map $z \mapsto \mathrm{Stab}(z)$ defines a μ-boundary η on $\mathrm{Sub}_{\mathrm{am}}(G)$. If η is not a Dirac mass on a finite normal subgroup of G, Theorem J shows that in this case there exists an amenable normalish subgroup of G, contradicting the hypothesis. Thus $\eta = \delta_{\{e_G\}}$. □

Examples of groups with no amenable normalish subgroups include groups with some positive ℓ^2-Betti number [BFS14, Corollary 1.5], groups with some non-trivial bounded cohomology with mixing coefficients, and linear groups with a finite amenable radical [BKKO17, Propositions 6.3 & 6.4]. The class of groups G such that $H^2_{\mathrm{b}}(G, \ell^2(G))$ is non-trivial is denoted by $\mathcal{C}_{\mathrm{reg}}$ in [MS06], and the groups in $\mathcal{C}_{\mathrm{reg}}$ do not admit amenable normalish subgroups by the previous sentence. The class $\mathcal{C}_{\mathrm{reg}}$ is known to contain

- countable groups admitting non-elementary, isometric and proper actions on simplicial trees, proper CAT(-1) spaces or Gromov-hyperbolic graphs of bounded valency [MS06, Theorem 1.3], [MS04, Corollaries 7.6 & 7.8, Theorem 7.13], [MMS04, Theorem 3],
- acylindrically hyperbolic groups [Osi16, Corollary 1.5], and
- countable groups admitting a non-elementary, metrically proper and essential action on a finite-dimensional irreducible CAT(0) cube complex [CFI16, Corollary 1.8].

Remarks.

- Theorem J should be compared with the results of [HK23, Section 6], where several classes of groups exhibiting non-positive curvature are shown to not possess any probability measure μ with a non-trivial amenable μ-SRS. Their conclusions are stronger, as they rule out the existence of any non-trivial μ-stationary measure on $\mathrm{Sub}_{\mathrm{am}}(G)$, not only those that arise as μ-boundaries of G. On the other hand, the class of groups covered by Theorem J is broader, and the proof is based on an argument that applies uniformly to all groups appearing in the statement.

- Following [KK17] and [HK23], A. Alpeev showed that a countable group G is C*-simple if and only if for a Baire-generic probability measure in $\mathrm{Prob}(G)$, the group G acts essentially freely on the associated Poisson boundary [Alp25]. Corollary K ensures hypotheses under which every probability measure has this property, instead of just a Baire-generic subset.

Finally, in Section 5.4 we observe that one cannot expect a converse to Theorem J. More precisely, we show that many Baumslag-Solitar groups have amenable normalish subgroups but their space of amenable subgroups is countable. Thus they admit no non-trivial amenable μ-SRSs for any non-degenerate μ, see Corollary 5.4.1. This is the case for $\mathrm{BS}(2,3) = \langle a, t \mid ta^2t^{-1} = a^3 \rangle$, for instance. The proof uses an unpublished result of Y. Cornulier [Cor].

5.1.1 Organization of the chapter

In Section 5.2 we recall basic properties of boundary actions of groups as well as their connections with C*-simplicity. We also discuss our results and further questions. Section 5.3 deals with proofs showing the existence of boundary SRSs and in particular with the proof of Theorem H. Finally, in Section 5.4 we prove Theorem J and show that many Baumslag-Solitar groups have amenable normalish subgroups but no amenable μ-SRSs.

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5.2 Background and discussion

We give some background on random walks, stationary boundaries and the space of subgroups of a group. We refer to [Fur02, LBMB18] for more details on this material. We also discuss C*-simple measures and further questions. In this section G is always a countable group.

5.2.1 Random walks and stationary spaces

Let μ be a probability measure on G. The *Shannon entropy* of μ is the non-negative quantity $H(\mu) = -\sum_{g\in G} \mu(g) \log \mu(g)$. Let $\mathbb{P}$ be the law of the right random walk $\mathbf{w} = (w_n)_{n\geq 0} \in G^{\mathbb{N}}$ driven by μ, that is, $w_n = g_1 \cdots g_n$ where $(g_n)_{n\geq 0} \subseteq G$ is a sequence of i.i.d. random variables with distribution μ. Denote by $\sigma\colon G^{\mathbb{N}} \to G^{\mathbb{N}}$ the shift $\sigma((w_n)_{n\geq 0}) = (w_{n+1})_{n\geq 0}$. We equip $G^{\mathbb{N}}$ with the action of G by multiplication on the left, so $g.(w_n)_{n\geq 0} = (gw_n)_{n\geq 0}$ for all $g \in G$, $(w_n)_{n\geq 0} \in G^{\mathbb{N}}$.

Let X be a compact metric space on which G acts by homeomorphisms. A Borel probability measure η on X is *μ-stationary* if $\eta = \sum_{g\in G} \mu(g) g_* \eta$. The martingale convergence theorem ensures that $\mathbb{P}_\mu$-almost surely the limit $\lim_{n\to\infty} (w_n)_* \eta = \eta_{\mathbf{w}}$ exists in $\mathrm{Prob}(X)$. When $\eta_{\mathbf{w}}$ is $\mathbb{P}_\mu$-almost surely a Dirac mass, we call the probability space (X, η) a *μ-boundary* of G. In this case, we obtain a G-equivariant and S-invariant map $(G^{\mathbb{N}}, \mathbb{P}_\mu) \to (X, \eta)$, where $S\colon G^{\mathbb{N}} \to G^{\mathbb{N}}$ is the shift map $S((w_n)_{n\geq 0}) = (w_{n+1})_{n\geq 0}$. Conversely, if there is $x \in X$ such that $\mathbb{P}_\mu$-almost surely $\lim_{n\to\infty} w_n x$ exists in X, the distribution of $\lim_{n\to\infty} w_n x$ is μ-stationary and defines a μ-boundary on X. The literature sometimes refers to μ-boundaries as *μ-proximal systems* [FG10].

By abuse of notation, a probability space (X, η) where G acts by nonsingular measurable transformations is also called a μ-boundary if it is μ-stationary and admits a compact metrizable model which is a μ-boundary in the above sense. The *Poisson boundary* of (G, μ) is the maximal μ-boundary of G, in the sense that any other such space is a G-equivariant quotient of it. It is uniquely defined up to G-equivariant measurable isomorphisms.

We will use the following well-known result, which is a consequence of the ergodicity of μ-boundaries.

Proposition 5.2.1. *Let η be a μ-boundary on a compact metric space X. If either η has atoms or η is G-invariant, then η is a Dirac mass on a point of X fixed by the action of G.*

Proof. Suppose first that η is G-invariant. Since $\mathbb{P}_\mu$-almost surely $\lim_{n\to\infty} (w_n)_* \eta = \delta_{\xi(\mathbf{w})}$ for some $\xi(\mathbf{w}) \in X$, we deduce that η is a Dirac mass on a fixed point of the action of G on X.

Now suppose η has atoms. By considering an atom of maximal η-mass, we deduce that η gives positive mass to a finite G-orbit $\mathcal{O}$. The ergodicity of η (see [Jaw94, Section 2], for instance) implies that it gives full mass to $\mathcal{O}$ and is thus invariant. By the previous paragraph, η must be a Dirac mass on a fixed point. □

5.2.2 The Chabauty space

By identifying subgroups of G with their indicator functions on G we endow $\mathrm{Sub}(G)$, the space of all subgroups G, with the topology inherited from $\{0,1\}^G$. This topology is known as the *Chabauty topology* and turns $\mathrm{Sub}(G)$ into a compact metrizable space with a basis composed of the sets

$$\{H \in \mathrm{Sub}(G) : Q \cap H = Q \cap K\}$$

for $K \in \mathrm{Sub}(G)$ and finite $Q \subseteq G$ [Cha50]. The action of G on itself by conjugation induces an action on $\mathrm{Sub}(G)$ by homeomorphisms. We denote by $\mathrm{Sub}_{\mathrm{am}}(G)$ the G-invariant subset of

Sub(G) of all amenable subgroups of G. Since a subgroup of G is non-amenable if and only if it contains a finitely generated non-amenable group, $\mathrm{Sub}_{\mathrm{am}}(G)$ is closed in Sub(G).

5.2.3 Hartman-Kalantar's characterizations of C*-simplicity

Denote the left-regular representation of the group G by $\lambda : G \to \mathcal{U}(\ell^2(G))$. Recall that a *state* on the reduced C*-algebra $C_r^*(G)$ is a positive linear functional $\rho : C_r^*(G) \to \mathbb{C}$ such that $\rho(\lambda_{e_G}) = 1$. The *canonical trace* τ_0 is the state on $C_r^*(G)$ that satisfies $\tau_0(\lambda_g) = 0$ for all $g \in G \setminus \{e_G\}$. The group G acts on a state ρ as $(g\rho)(a) = \rho(\lambda_{g^{-1}ag})$ for each $g \in G$ and, given a probability measure μ on G, the state ρ is called μ-stationary if $\rho = \sum_{g \in G} \mu(g) g\rho$. Since the canonical trace is invariant under the action of G, it is in particular μ-stationary for every $\mu \in \mathrm{Prob}(G)$.

Hartman-Kalantar define a *C*-simple measure* as a probability measure μ on G such that the canonical trace is the unique μ-stationary state on $C_r^*(G)$ [HK23, Definition at the top of page 4]. Their first main theorem states that a countable group G is C*-simple if and only if G possesses a C*-simple probability measure [HK23, Theorem 5.2]. Their second main theorem shows that a countable group G is C*-simple if and only if G admits a probability measure μ for which the space $\mathrm{Sub}_{\mathrm{am}}(G)$ of amenable subgroups is uniquely μ-stationary [HK23, Corollary 5.7]. The following implication between these two notions of unique stationarity is proved and used in [HK23].

Proposition 5.2.2. *Let μ be a probability measure on a countable group G. Suppose that $\mathrm{Sub}_{\mathrm{am}}(G)$ admits a μ-SRS distinct from $\delta_{\{e_G\}}$. Then the space of states on the reduced C*-algebra $C_r^*(G)$ is not uniquely μ-stationary.*

Proof. Let η be an amenable μ-SRS distinct from $\delta_{\{e_G\}}$. By [HK23, Lemma 2.3], the map

$$\rho(\lambda_g) \coloneqq \eta\left(\{H \in \mathrm{Sub}_{\mathrm{am}}(G) : g \in H\}\right), \ g \in G,$$

extends to a state ρ on $C_r^*(G)$. Furthermore, since η is μ-stationary and distinct from $\delta_{\{e_G\}}$, the state ρ is μ-stationary and it differs from the canonical trace. □

Our criterion in Theorem H for the existence of probability measures $\mu \in \mathrm{Prob}(G)$ such that $\mathrm{Sub}_{\mathrm{am}}(G)$ is not uniquely μ-stationary therefore shows that these measures are not C*-simple in the sense of Hartman–Kalantar. However, we remark that we do not know whether the converse to Proposition 5.2.2 is true. In principle, there may exist probability measures μ on G that are not C*-simple even though $\mathrm{Sub}_{\mathrm{am}}(G)$ is uniquely μ-stationary.

5.2.4 SRSs that are not μ-boundaries

Let μ be a probability measure on a countable group G. All results in this chapter concern amenable μ-stationary random subgroups for which the space $(\mathrm{Sub}_{\mathrm{am}}(G), \eta)$ is a μ-boundary (see Definition 5.1.2). It follows from work of H. Furstenberg [Fur73, Theorem 14.1] and Glasner-Weiss [GW16, Theorem 8.5] that, if G acts by homeomorphisms on a compact metric space X, then the following conditions are equivalent:

- For every μ-stationary probability measure ν on X, the space (X, ν) is a μ-boundary.
- There exists a unique μ-stationary probability measure ν on X, and the space (X, ν) is a μ-boundary.

In particular, when $X = \mathrm{Sub}_{\mathrm{am}}(G)$, Question 5.1.1 asks for probability measures μ on G such that $\mathrm{Sub}_{\mathrm{am}}(G)$ is not uniquely μ-stationary. The equivalence above shows that, in this setting, our methods cannot identify all amenable μ-stationary random subgroups, since there will necessarily exist some that do not arise as μ-boundaries of G. This leads to the natural question of whether Theorem J extends to arbitrary stationary random subgroups of G.

Question 5.2.3. Let μ be non-degenerate probability measure on a countable group G, and let η be an ergodic μ-stationary probability measure on $\mathrm{Sub}(G)$ that is not a Dirac mass on a finite normal subgroup of G.

- Is it true that η must be supported on normalish subgroups of G?
- Suppose further that η is supported on amenable subgroups of G. Is it true that all ℓ^2-Betti numbers of G must vanish?

We remark that if η is an IRS of a countable group G supported on subgroups whose ℓ^2-Betti numbers all vanish, then it is known that the same conclusion holds for G. Indeed, as shown by Abert-Glasner-Virag, every IRS of G arises as the image of a p.m.p. action of G under the stabilizer map [AGV14, Proposition 13]. This gives rise to the "principal extension" of groupoids in the language of [ST10, Section 5.3]. It follows from [ST10, Lemma 5.1] that the ℓ^2-Betti numbers of the stabilizer groupoid vanish. Then [ST10, Theorem 1.3] shows that the ℓ^2-Betti numbers of the groupoid of the action $G \curvearrowright (\mathrm{Sub}(G), \eta)$ vanish as well, and these coincide with the ℓ^2-Betti numbers of G (see [CGdlS21, Theorem 5.4] for instance). We are grateful with Sam Mellick and Damien Gaboriau for bringing this argument to our attention.

5.3 Existence of amenable boundary μ-SRSs

This section presents results on the existence of non-trivial amenable boundary stationary random subgroups in countable groups. We begin with a short proof for wreath products (Proposition 5.3.2), which also serves as motivation for the proof of Theorem H, where we establish a general criterion guaranteeing their existence. We then apply this criterion to Thompson's group F (Corollary I) and to permutational wreath products (Corollary 5.3.8).

5.3.1 A short proof for wreath products

Let A, B be countable groups. Their wreath product is $A \wr B = \bigoplus_B A \rtimes B$, where $\bigoplus_B A$ denotes the group of finitely supported functions $\varphi : B \to A$, and the action of B on $\bigoplus_B A$ is given by

$$(x \cdot \varphi)(y) = \varphi(x^{-1}y), \qquad x, y \in B.$$

Given a probability measure $\mu \in \operatorname{Prob}(A \wr B)$, the μ-random walk $\mathbf{w} = (w_n)_{n\geq 0}$ on $A \wr B$ can be written using the semidirect product structure as $w_n = (\varphi_n, b_n)$. Here $\varphi_n \in \bigoplus_B A$ is called the *lamp configuration at time* n, and $b_n \in B$ is the *position at time* n. In general $(\varphi_n)_{n\geq 0}$ does not define a random walk on $\bigoplus_B A$, whereas $(b_n)_{n\geq 0}$ is a random walk on the base group B.

Definition 5.3.1. *We say that the lamp configurations of the μ-random walk on $A \wr B$* stabilize almost surely *if there exists a conull set of trajectories $\mathbf{w} = (\varphi_n, b_n)_{n\geq 0}$ such that for every $b \in B$ there exists N for which $\varphi_n(b) = \varphi_N(b)$ for all $n \geq N$.*

Under this assumption, there is a $\mathbb{P}_\mu$-almost everywhere defined map $(A \wr B)^{\mathbb{N}} \to A^B$ which assigns to a trajectory $\mathbf{w} = (\varphi_n, b_n)_{n\geq 0}$ the *limit lamp configuration* $\varphi_\infty(\mathbf{w})$ defined by

$$\varphi_\infty(\mathbf{w})(b) = \lim_{n\to\infty} \varphi_n(b), \qquad b \in B.$$

Lamp configurations do not always stabilize: for every $\varepsilon > 0$ there exist probability measures μ with finite $(1-\varepsilon)$-moment for which stabilization fails [Kai83]. However, a sufficient condition for stabilization is that μ has finite first moment and the induced random walk on B is transient [Ers11]. If B is finitely generated with at least cubic growth, then a theorem of Varopoulos [Var86] implies that every non-degenerate probability measure on $A \wr B$ induces a transient random walk on B. In particular, in this case one may choose μ to be finitely supported.

Proposition 5.3.2. *Let A, B be countable groups, and let μ be a non-degenerate probability measure on $A \wr B$ such that the lamp configurations of the μ-random walk on $A \wr B$ stabilize almost surely. Then $\operatorname{Sub}_{\mathrm{ab}}(A \wr B)$ supports a μ-boundary distinct from $\delta_{\{e_G\}}$, which may be taken non-atomic if A is non-abelian.*

Proof. For $b \in B$ and $a \in A$, define $\delta_a^b \in \bigoplus_B A \subseteq A \wr B$ as the configuration equal to a at b and to e_A elsewhere.

Let us fix an arbitrary $a \in A \setminus \{e_A\}$. Given a trajectory $\mathbf{w} = (\varphi_n, b_n)_{n\geq 0} \in (A \wr B)^{\mathbb{N}}$ with limit lamp configuration $\varphi_\infty(\mathbf{w}) = (\varphi_\infty(\mathbf{w})_b)_{b\in B}$, let $H(\mathbf{w})$ be the subgroup of $\bigoplus_B A$ defined as $H(\mathbf{w}) = \langle \delta^b_{\varphi_\infty(\mathbf{w})_b . a} : b \in B \rangle$.

Since the map $\mathbf{w} \mapsto \varphi_\infty(\mathbf{w})$ is $(A \wr B)$-equivariant, we have for every $g = (\varphi, b') \in A \wr B$ the equalities

$$\begin{aligned} H(g.\mathbf{w}) =& \langle \delta^b_{(g.\varphi_\infty(\mathbf{w}))_b . a} : b \in B \rangle = \langle \delta^b_{(\varphi_b \varphi_\infty(\mathbf{w})_{b'^{-1}b}).a} : b \in B \rangle \\ =& \langle \delta^{b'b}_{(\varphi_{b'b} \varphi_\infty(\mathbf{w})_b).a} : b \in B \rangle = g.H(\mathbf{w}). \end{aligned}$$

Thus the push-forward η of $\mathbb{P}_\mu$ through $\mathbf{w} \mapsto H(\mathbf{w})$ is a μ-boundary supported on $\operatorname{Sub}_{\mathrm{ab}}(A \wr B)$ distinct from $\delta_{\{e_G\}}$. If A is non-abelian, we can pick a not in the center of A, so that η is non-atomic. ☐

The group $A \wr B$ is known to be C*-simple whenever A is C*-simple; see [BO18, Proposition 5.5] for example. Alternatively, using [KK17, Theorem 1.5], one can prove the C*-simplicity of $A \wr B$ by exhibiting a topologically free $(A \wr B)$-boundary. That is, it suffices to show the existence of a compact space X equipped with a minimal action of $A \wr B$ by homeomorphisms such that the closure of every $(A \wr B)$-orbit in $\mathrm{Prob}(X)$ contains a Dirac delta measure, and such that the set of fixed points in X of every non-trivial element of $A \wr B$ has an empty interior. In order to do so, assume that A is C*-simple and let X_A be a topologically free A-boundary. Consider the action of $A \wr B$ on $\prod_B X$ given by $(\varphi, c) \cdot (x_b)_{b \in B} = (\varphi_b \cdot x_{c^{-1}b})_{b \in B}$ for every $\varphi \in \bigoplus_B A$, $c \in B$ and $(x_b)_{b \in B} \in \prod_B X_A$. This yields a topologically free boundary action of $A \wr B$, and hence $A \wr B$ is C*-simple.

As a consequence of Proposition 5.3.2, we obtain the following.

Corollary 5.3.3. *Let A be a countable C*-simple group and let B be a finitely generated group of at least cubic growth. Then every C*-simple probability measure on $A \wr B$, in the sense of [HK23], has infinite first moment.*

5.3.2 Proof of Theorem H

For the construction of the measure appearing in Theorem H we first need to introduce some general terminology.

Definition 5.3.4. *Let $p = (p_j)_{j \geq 0}$ be a probability measure on $\mathbb{N}$, and let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with law p. For every $n \in \mathbb{N}_{>0}$ let us denote by $M_n := \max\{X_1, X_2, \dots, X_n\}$ the* record value of the sequence at time *n. We say that the record value M_n is* simple *if*

$$|\{i \in \{1, \dots, n\} : X_i = M_n\}| = 1.$$

We say that the records of $(X_n)_{n \geq 1}$ are eventually simple *if $p^{\otimes \mathbb{N}}$-almost surely there is $N \geq 1$ such that, for all $n \geq N$, the record value M_n is simple. Finally, define the sequence $(T_k)_{k \geq 1}$ of* record times of $(X_n)_{n \geq 1}$ *by $T_1 = 1$ and $T_{k+1} = \min\{n > T_k : X_n = M_n\}$ for $k \in \mathbb{N}_{>0}$.*

The following lemma can be found in [EK23, Lemma 2.3 & Corollary 2.6]. The sufficient condition is due to [BSW94, Theorem 3.2], while the necessary condition is due to [Qi97, Theorem 2]. See also the proof given in [Eis09, Theorem 3, Corollaries 3.1 & 3.2].

Lemma 5.3.5. *Let p be a probability measure on $\mathbb{N}$ and let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with law p. Then $(X_n)_{n \geq 1}$ has eventually simple records if and only if p has an infinite support and*

$$\sum_{j=0}^{\infty} \left(\frac{p_j}{p_j + p_{j+1} + \cdots} \right)^2 < \infty. \tag{5.3.1}$$

The latter condition holds in particular when p_j has polynomial decay as $j \to \infty$.

Lemma 5.3.6 ([EK23, Lemma 2.17])**.** *For any probability measure p on $\mathbb{N}$ with infinite support there exists a non-decreasing function $\Phi \colon \mathbb{N} \to \mathbb{N}$ such that almost surely $T_{k+1} \leq \Phi(R_k)$ holds for all sufficiently large k.*

Proposition 5.3.7. *Let G be a countable group. Assume that there exists a non-trivial subgroup H such that, for all finite subsets $Q, Z \subseteq G$ there is $b \in G$ such that bZ and $b^{-1}Z$ are contained in*

$$\{g \in G : Q \cap H = Q \cap gHg^{-1}\}.$$

Then there exists a non-degenerate, symmetric and finite-entropy probability measure μ on G such that $\mathbb{P}_\mu$-almost surely the sequence $(w_n H w_n^{-1})_{n\geq 0}$ converges to a non-trivial subgroup of G.

In particular, the orbit $\overline{\mathrm{Orb}_G(H)}$ supports a μ-boundary distinct from $\delta_{\{e_G\}}$.

Proof. Consider a probability measure p on $\mathbb{N}$ that gives positive mass to all non-negative integers and that satisfies Condition (5.3.1), so that Lemma 5.3.5 ensures that a sequence of i.i.d. random variables with law p almost surely has eventually simple records. Moreover, we may choose p such that its Shannon entropy $H(p) = -\sum_{j\geq 0} p_j \log(p_j)$ is finite.

Let us choose a sequence $(\widetilde{A}_n)_{n\geq 0}$ of pairwise disjoint and symmetric subsets of G such that $\widetilde{A}_0 = \{e_G\}$, $|\widetilde{A}_n| \leq 2$ for all $n \geq 0$ and $G = \bigcup_{n\geq 0} \widetilde{A}_n$. We start by constructing inductively increasing sequences of finite subsets $(A_n)_{n\geq 0}$, $(\triangle_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$ of G, along with a sequence of elements $b_n \in G$ for every $n \geq 0$.

We first set $A_0 = \widetilde{A}_0 = \{e_G\}$, $\triangle_0 = Q_0 = \varnothing$ and $b_0 = e_G$. Now, let $n \geq 1$ and suppose that the sets $\triangle_i$ and Q_i together with the element b_i have already been defined for every $0 \leq i \leq n$. We set

$$A_{n+1} = \widetilde{A}_{n+1} \setminus \bigcup_{i=0}^{n} \{b_i, b_i^{-1}\},\ \triangle_{n+1} = \left(\bigcup_{i=0}^{n} A_i \cup \{b_i, b_i^{-1}\}\right)^{\Phi(n)},$$

where Φ is the gauge function associated with p from Lemma 5.3.6. Next, we set Q_{n+1} to be a finite subset of G containing Q_n and such that for every $d \in \triangle_n$ we have $Q_{n+1} \cap dHd^{-1} \neq \{e_G\}$. Applying the hypothesis of the theorem to the finite subsets $Q := \bigcup_{d\in\triangle_n} d^{-1}Q_{n+1}d$ and $Z := \triangle_n$, we find $b_{n+1} \in G$ such that for all $z \in \triangle_n$ we have

$$Q \cap H = Q \cap (b_{n+1}z)H(b_{n+1}z)^{-1} = Q \cap (b_{n+1}^{-1}z)H(b_{n+1}^{-1}z)^{-1}.$$

As a consequence, notice that for each $d, z \in \triangle_n$ we have

$$Q_{n+1} \cap dHd^{-1} = Q_{n+1} \cap (db_{n+1}z)H(db_{n+1}z)^{-1} = Q_{n+1} \cap (db_{n+1}^{-1}z)H(db_{n+1}^{-1}z)^{-1}. \tag{5.3.2}$$

This completes the inductive construction of the sequences $(A_n)_{n\geq 0}$, $(\triangle_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$.

Let us now fix an auxiliary sequence $(\alpha_i)_{i\geq 1}$ with $0 < \alpha_i < 1$ for every $i \geq 1$ and such that $\sum_{i\geq 1} \alpha_i < \infty$. Define a symmetric probability measure μ on G with $\mu(A_0) = p_0$ and such that for every $i \geq 1$ we have

$$\mu\left(A_i \cup \{b_i, b_i^{-1}\}\right) = p_i \text{ and } \mu\left(A_i \setminus \{b_i, b_i^{-1}\}\right) \leq \alpha_i p_i. \tag{5.3.3}$$

In what remains of the proof, we will justify that such a measure μ satisfies the conclusions of the theorem.

First, notice that $\mathrm{supp}(\mu) = G$ because p is fully supported and $G = \cup_{n\geq 0} A_n$. In particular, the probability measure μ is non-degenerate. Moreover, μ has finite Shannon entropy since

$$H(\mu) \leq \log(4) + H(p) < \infty. \tag{5.3.4}$$

Indeed, to see this it suffices to notice that μ can be sampled in two steps: first, choose an index $i \in \mathbb{N}$ according to the distribution p, then sample $g_i \in A_i \cup \{b_i, b_i^{-1}\}$ according the conditional distribution of μ on that subset. Inequality (5.3.4) then follows from the standard (conditional) entropy decomposition $H(X) = H(X \mid Y) + H(Y)$ for random variables X and Y, together with the fact that the entropy of a random variable taking values in a set of size N is at most $\log(N)$. Since $|A_i \cup \{b_i, b_i^{-1}\}| \leq 4$ for every $i \geq 0$, we obtain the desired bound.

It now remains to show the $\mathbb{P}_\mu$-almost sure convergence of the sequence $(w_n H w_n^{-1})_{n\geq 0}$ to a non-trivial subgroup of G. Let $(g_n)_{n\geq 1}$ be an independent sequence of elements of G distributed according to μ and denote $w_n = g_1 \cdots g_n$ for every $n \geq 1$. In other words, the process $(w_n)_{n\geq 1}$ has the law $\mathbb{P}_\mu$ of the μ-random walk on G. For every $i \geq 1$, denote $X_i \in \mathbb{N}$ the unique value such that $g_i \in A_{X_i} \cup \{b_{X_i}, b_{X_i}^{-1}\}$. Then, it follows from the construction of μ that the sequence $(X_i)_{i\geq 1}$ is i.i.d. with law p, and hence has eventually simple records. Denote by $(T_k)_{k\geq 1}$ the associated record times and by $R_k \coloneqq X_{T_k}$ the record values for every $k \geq 1$. By combining

(a) that $\mu(A_i \backslash \{b_i, b_i^{-1}\}) \leq \alpha_i p_i$ (Condition (5.3.3)) with $\sum_{i\geq 1} \alpha_i < \infty$ and the Borel-Cantelli lemma,

(b) Lemma 5.3.6, and

(c) the definition of eventually simple record times (Definition 5.3.4),

we conclude that almost surely there exists $k_0 \geq 1$ such that for all $k \geq k_0$ we have

(A) $g_{T_k} \in \{b_{R_k}, b_{R_k}^{-1}\}$,

(B) $T_{k+1} \leq \Phi(R_k)$, and

(C) $X_m < R_k$ for all $T_k < m < T_{k+1}$.

Now, let $n > T_{k_0}$ and choose $k \geq k_0$ such that $T_k \leq n < T_{k+1}$. Then, we can write

$$w_n = g_1 \cdots g_{T_k - 1} \cdot g_{T_k} \cdot g_{T_k+1} \cdots g_n.$$

Set $d = g_1 \cdots g_{T_k-1}$ and $z = g_{T_k+1} \cdots g_n$, so that $w_n = d g_{T_k} z$. It follows from Conditions (B) and (C) above that

$$d, z \in \left(\bigcup_{i=0}^{R_k - 1} A_i \cup \{b_i, b_i^{-1}\} \right)^{T_{k+1}} \subseteq \triangle_{R_k}.$$

In addition, thanks to Condition (A) together with the inductive construction of b_{R_k}, we also have that

$$Q_{R_k} \cap dHd^{-1} = Q_{R_k} w_n H w_n^{-1},$$

as in Equation (5.3.2).

Claim. *For every $m \geq n$, we have*

$$Q_{R_k} \cap dHd^{-1} = Q_{R_k} \cap w_m H w_m^{-1}.$$

Proof of the claim. Let $l \in \mathbb{N}$ be such that $T_l \leq m < T_{l+1}$. Notice that since $m > n$, we must have $l \geq k$. We prove the claim by induction on l.

If $l = k$, then the claim was already shown to hold in Equation (5.3.2). Now suppose that $l > k$ and write $w_m = \widetilde{d} g_{T_l} \widetilde{z}$, where

$$\widetilde{d} = g_1 \cdots g_{T_l - 1} \text{ and } \widetilde{z} = g_{T_l+1} \cdots g_m.$$

As in the argument preceding the claim, we have

$$Q_{R_l} \cap \widetilde{d} H \widetilde{d}^{-1} = Q_{R_l} \cap w_m H w_m^{-1},$$

thus in particular

$$Q_{R_k} \cap \widetilde{d} H \widetilde{d}^{-1} = Q_{R_k} \cap w_m H w_m^{-1}. \tag{5.3.5}$$

Applying the inductive hypothesis with $\widetilde{d} = w_{T_l - 1}$ gives

$$Q_{R_k} \cap d H d^{-1} = Q_{R_k} \cap \widetilde{d} H \widetilde{d}^{-1}. \tag{5.3.6}$$

Combining Equations (5.3.5) and (5.3.6) concludes the proof of the claim. ☐

The compactness of $\mathrm{Sub}(G)$ implies that the sequence $(w_n H w_n^{-1})_{n \geq 0}$ has at least one limit point $K \in \mathrm{Sub}(G)$. Since the sets Q_{R_k} form an exhaustion of G, the previous claim shows that, for every open neighborhood U of K, the sequence $(w_n H w_n^{-1})_{n \geq 0}$ eventually remains in U. Hence the limit point is unique, and $(w_n H w_n^{-1})_{n \geq 0}$ converges in $\mathrm{Sub}(G)$. Finally, the limit subgroup K is non-trivial: indeed, $Q_{R_k} \cap d H d^{-1}$ contains a non-trivial element $g \in G \backslash \{e_G\}$, and the previous claim shows that $g \in K$. ☐

Remark. Theorem H requires the existence of a non-trivial subgroup $H \leq G$ such that for every pair of finite subsets $Q, Z \subseteq G$ there exists $b \in G$ with

$$bZ,\, b^{-1}Z \subseteq \{g \in G : Q \cap H = Q \cap gHg^{-1}\}.$$

This condition admits a dynamical interpretation for the conjugation action of G on $\mathrm{Sub}(G)$. The finite set Q determines a basic neighbourhood of H in the Chabauty topology by

$$U_Q(H) = \{K \in \mathrm{Sub}(G) : Q \cap K = Q \cap H\}.$$

The above property asserts that, for every such neighbourhood and every finite set $Z \subseteq G$, there exists $b \in G$ such that

$$bz \cdot H,\, b^{-1}z \cdot H \in U_Q(H) \qquad \text{for all } z \in Z.$$

In other words, any finite portion of the G-orbit of H in $\mathrm{Sub}(G)$ can be simultaneously pushed arbitrarily close to H by conjugation with a single group element b as well as by its inverse. This is a strong form of recurrence for the action of G on the orbit of H.

5.3.3 Amenable boundary μ-SRSs for Thompson's group F

Consider the group of orientation-preserving homeomorphisms $f\colon \mathbb{R} \to \mathbb{R}$ such that there exists a finite subset $D \subset \mathbb{Z}\left[\frac{1}{2}\right]$ with the property that for each bounded connected component C of $\mathbb{R} \setminus D$, the map f restricted to C is of the form

- $x \mapsto 2^k x + q$ where $k \in \mathbb{Z}$ and $q \in \mathbb{Z}[1/2]$ if C is bounded, and of the form
- $x \mapsto x + m$ where $m \in \mathbb{Z}$ if C is unbounded.

It is well known that the above describes a group isomorphic to Thompson's group F, the group of all increasing piecewise dyadically affine homeomorphisms of the closed unit interval $[0, 1]$; see, e.g., [BB05, Section 3.1]. In this representation, the commutator subgroup $[F, F]$ coincides with the homeomorphisms in F which are the identity outside of a compact interval of $\mathbb{R}$.

Proof of Corollary I. We show that Thompson's group F satisfies the hypotheses of Theorem H. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary non-trivial element of F whose support is contained in a proper closed subinterval of $[0, 1]$. For each $k \in \mathbb{Z}$ denote by $t_k \in F$ the translation by $k \in \mathbb{Z}$. Let us consider the subgroup H of F generated by all elements $t_k f t_k^{-1}$, $k \in \mathbb{Z}$. These elements commute since they have disjoint supports, so H is abelian. Notice also that H and all of its conjugates are contained in the normal subgroup $[F, F]$. Hence, to verify the condition of Theorem H, it suffices to show that for any finite subset $Q \subseteq [F, F]$ and every finite subset $Z \subseteq F$ we can find $b \in F$ such that

$$bZ, b^{-1}Z \subseteq \{g \in F : Q \cap H = Q \cap gHg^{-1}\}.$$

Since $Q \subseteq [F, F]$, all elements of Q are the identity outside of a sufficiently large interval. Thus we can choose $N \in \mathbb{N}$ such that $\operatorname{supp}(q) \subseteq [-N, N]$ for every $q \in Q$. Next, choose $M \geq 1$ large enough so that for every $z \in Z$ there exist $m_z^-, m_z^+ \in \mathbb{Z}$ such that

$$z(x) = x + m_z^- \text{ for } x \leq -M, \;\; z(x) = x + m_z^+ \text{ for } x \geq M.$$

For every $z \in Z$, denote by $r_z^-, r_z^+ \in F$ the maps $r_z^-(x) = x - m_z^-$ and $r_z^+(x) = x - m_z^+$ for $x \in \mathbb{R}$. Then $\operatorname{supp}(zr_z^-) \subseteq [-M, +\infty)$ and $\operatorname{supp}(zr_z^+) \subseteq (-\infty, M]$ for each $z \in Z$.

Now let $b \in F$ be the translation by $-M - N - 1$. Using the fact that H is normalized by any translation, for every $z \in Z$ we have

$$\begin{aligned} bzHz^{-1}b^{-1} &= b(zr_z^+)H((r_z^+)^{-1}z^{-1})b^{-1} \\ &= (b(zr_z^+)b^{-1})H(b(zr_z^+)b^{-1})^{-1}. \end{aligned}$$

The support of $b(zr_z^+)b^{-1}$ is contained in $(-\infty, -N-1]$, while all elements in Q have support contained in $[-N, N]$. Therefore, $b(zr_z^+)b^{-1}$ commutes with all elements in Q, implying that bZ is contained in $\{g \in F : Q \cap H = Q \cap gHg^{-1}\}$. An analogous argument applies to $b^{-1}Z$. This verifies the hypotheses of Theorem H, and the conclusion follows from the fact that the set of abelian subgroups of $\operatorname{Sub}(F)$ is closed in the Chabauty topology. $\square$

5.3.4 Amenable boundary μ-SRSs for permutational wreath products

Let A, B be countable groups and let $B \curvearrowright X$ be an action on a countable set X. The *permutational wreath product* $G = A \wr_X B$ is the semidirect product $\left(\bigoplus_{x \in X} A\right) \rtimes B$, where B acts on $\bigoplus_{x \in X} A$ by $b.(\varphi_x)_{x \in X} = (\varphi_{b^{-1}.x}))_{x \in X}$ for $b \in B$, $(\varphi_x)_{x \in X} \in \bigoplus_{x \in X} B$. Here, the group B is identified with its canonical copy inside $A \wr_X B$, whereas A is identified with its copy within the direct sum at a given fixed basepoint $o \in B$.

Let $\mu \in \operatorname{Prob}(A \wr_X B)$. As in the case of wreath products, we say that the lamp configurations of the μ-random walk on $A \wr_X B$ stabilize almost surely if there is an infinite orbit $\mathcal{O} \subseteq X$ such that for $\mathbb{P}_\mu$-almost every trajectory $\mathbf{w} = (\varphi_n, b_n)_{n \geq 0}$ the configurations $(\varphi_n)_{n \geq 0}$ restricted to $\mathcal{O}$ converge pointwise to a limit function $\varphi_\infty(\mathbf{w}) \in A^{\mathcal{O}}$. Stabilization of lamps occurs whenever the induced random walk $(b_n.x)_{n \geq 0}$ on $\mathcal{O}$ is transient for every $x \in \mathcal{O}$ and μ has finite first moment. In general, transience of $(b_n.x)_{n \geq 0}$ on $\mathcal{O}$ can be more delicate than transience of the random walk on $(b_n)_{n \geq 0}$ on B itself. The following proposition applies the criterion in Theorem H to construct a symmetric measure μ with finite entropy such that this is the case.

Corollary 5.3.8. *Let A, B be countable groups and let $B \curvearrowright X$ be a faithful action on a countable set X with at least one infinite orbit $\mathcal{O}$. Suppose that A is not abelian. Then there exists a non-degenerate and symmetric probability measure on $A \wr_X B$ with finite Shannon entropy that admits an amenable boundary μ-SRS distinct from $\delta_{\{e\}}$.*

Proof. Let us fix an element $a \in A$ that is not in the center of A, and consider the non-trivial amenable subgroup $H = \bigoplus_{x \in \mathcal{O}} \langle a \rangle$ of $A \wr_X B$. We will show that H satisfies the hypotheses of Theorem H.

First, we note that H is normalized by both subgroups $\bigoplus_{x \in X \setminus \mathcal{O}} A$ and B. Furthermore, H is contained in the normal subgroup $\bigoplus_{x \in \mathcal{O}} A$. In order to verify the conditions of Theorem H, it suffices to show that for any finite subsets $Z, Q \subset \bigoplus_{x \in \mathcal{O}} A$ we can find $b \in B$ such that

$$bZ, b^{-1}Z \subseteq \left\{g \in A \wr_X B : Q \cap H = Q \cap gHg^{-1}\right\}.$$

Let $Z, Q \subset \bigoplus_{x \in \mathcal{O}} A$ be finite subsets. Denote by $F_Z, F_Q \subset \mathcal{O}$ the union of the supports of elements in Z, Q, respectively.

Claim. *There exists $b \in B \subseteq A \wr_X B$ such that both bF_Z and $b^{-1}F_Z$ are disjoint from F_Q.*

Proof of the claim. Let μ be a non-degenerate symmetric probability measure on B, and denote by $\mathbb{P}_\mu$ the law of the μ-random walk $(w_n)_{n \geq 0}$ on B. Since $\mathcal{O}$ is infinite, it follows from [CF25, Lemma 3.1] (see also [GMBT24, Claim 2.3], or [LBMB18, Lemma 3.2]) that

$$\lim_{n \to \infty} \mathbb{P}_\mu[w_n.x = y] = 0$$

for every $x \in F_Z$ and $y \in F_Q$. A union bound then gives

$$\lim_{n \to \infty} \mathbb{P}_\mu[w_n.F_Z \cap F_Q = \varnothing] = 1.$$

For every $n \geq 1$, the law of the n-th step of the random walk $w_n = g_1 g_2 \cdots g_n$ is μ^{*n}. Since μ is symmetric, this is the same distribution as $w_n^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$. It then follows that

$$\lim_{n\to\infty} \mathbb{P}_\mu[(w_n)^{-1} \cdot F_Z \cap F_Q = \varnothing] = 1.$$

Another application of a union bound shows the existence of $b \in B$ such that

$$b.F_Z \cap F_Q = b^{-1} \cdot F_Z \cap F_Q = \varnothing. \qquad \square$$

By an argument analogous to the one in the proof of Corollary I, for every $z \in Z$ we can write

$$bzHz^{-1}b^{-1} = (bzb^{-1})Hb(bzb^{-1})^{-1},$$

and similarly

$$b^{-1}zHz^{-1}b = (b^{-1}zb)H(b^{-1}zb)^{-1}.$$

By the previous claim, bzb^{-1} and $b^{-1}zb$ both commute with every element in Q, so that bZ and $b^{-1}Z$ are contained in $\{g \in A \wr_X B : Q \cap H = Q \cap gHg^{-1}\}$. Therefore, we conclude that there exists an amenable boundary μ-SRS distinct from $\delta_{\{e\}}$. $\square$

Remark. Although the analysis of random walks on permutational wreath products is more complicated, the presence of inverted orbits leads to more diverse geometric phenomena. For instance, certain classes of permutational wreath products furnish the first examples of finitely generated groups with intermediate growth with explicit asymptotic growth functions [BE12] and finitely generated groups of exponential growth for which every finitely supported probability measure has trivial Poisson boundary [BE17].

5.4 Normalish subgroups and boundary μ-SRSs

In this section we prove Theorem J, which relates normalish subgroups of a group G to boundary SRSs on $\mathrm{Sub}(G)$. We also show that many C*-simple Baumslag-Solitar groups have amenable normalish subgroups even though their space of amenable subgroups is countable, so all their amenable μ-SRSs are trivial. We finally show a different proof of triviality of amenable μ-boundaries for groups with property (CS), an operator-algebraic property of groups formulated in terms of their unitary representations which was introduced in [BKKO17].

5.4.1 Proof of Theorem J

Recall that the *FC-center* of a group G is the subgroup $\mathrm{FC}(G)$ composed of all elements of G with finite conjugacy class. An action of a group G on a compact Hausdorff space X by homeomorphisms is said to be *equicontinuous* if the image of G in $\mathrm{Homeo}(X)$ is relatively compact for the compact-open topology.

Proof of Theorem J. Let μ be a non-degenerate probability measure on G and let η be a μ-stationary probability measure on $\mathrm{Sub}(G)$ such that the space $(\mathrm{Sub}(G), \eta)$ is a μ-boundary of G. If η is atomic, then it must be a Dirac mass on an infinite normal subgroup, and the result follows. We therefore assume that η is non-atomic. Let $g_1, \dots, g_m \in G$. We will show that for every $\varepsilon > 0$ we have

$$\eta\left(\left\{H \in \mathrm{Sub}(G) : \bigcap_{i=1}^{m} g_i H g_i^{-1} \text{ is finite}\right\}\right) \leq \varepsilon.$$

Since ε is arbitrary, the above measure must be 0, and as G is countable it follows that η is supported on normalish subgroups of G.

Claim. *The measure η is not supported in the FC-center* $\mathrm{FC}(G)$*.*

Proof of the claim. Consider the closed subspace $\mathrm{Sub}(\mathrm{FC}(G))$, and notice that the G-orbit of any open set of the form

$$\{H \in \mathrm{Sub}(\mathrm{FC}(G)) : Q \cap H = Q \cap K\},\ Q \subseteq \mathrm{FC}(G) \text{ finite and } K \in \mathrm{Sub}(\mathrm{FC}(G))$$

is finite by the definition of $\mathrm{FC}(G)$. Since these sets constitute a basis of the topology of $\mathrm{Sub}(\mathrm{FC}(G))$, we conclude that the action of G on the clopen subsets of $\mathrm{Sub}(\mathrm{FC}(G))$ has only finite orbits. This implies that the action is equicontinuous: indeed, it is an odometer in the sense of [Cor14, Définition 2.1.3]. By [Cor14, Fait 2.1.4 (iii)] it follows that $\mathrm{Sub}(\mathrm{FC}(G))$ is a projective limit of actions of G on finite sets, so it must be equicontinuous. Hence any μ-stationary measure on $\mathrm{Sub}(\mathrm{FC}(G))$ is invariant by [FG10, Theorem 7.4]. Thus η cannot be supported on $\mathrm{FC}(G)$, since any μ-boundary which is also an invariant measure is a Dirac mass. □

Thus there exists $h \in G$ with infinite conjugacy class such that

$$c \coloneqq \eta(\{H \in \mathrm{Sub}(G) : h \in H\}) > 0.$$

Denote by $\mathbf{bnd}\colon G^{\mathbb{N}} \to \mathrm{Sub}(G)$ the boundary map associated with η. Given a trajectory $\mathbf{w} = (w_n)_{n \geq 0} \in G^{\mathbb{N}}$ of the μ-random walk on G, we can write

$$\eta(\{H \in \mathrm{Sub}(G) : h \in w_n H w_n^{-1}\}) = (w_n)_* \eta(\{H \in \mathrm{Sub}(G) : h \in H\})$$

for every $n \in \mathbb{N}$. The fact that $(\mathrm{Sub}(G), \eta)$ is a μ-boundary implies

$$\eta(\{H \in \mathrm{Sub}(G) : h \in w_n H w_n^{-1}\}) \xrightarrow[n \to \infty]{} \delta_{\mathbf{bnd}(\mathbf{w})}(\{H \in \mathrm{Sub}(G) : h \in H\})$$

for $\mathbb{P}_\mu$-almost every trajectory $\mathbf{w} \in G^{\mathbb{N}}$. In particular, we have

$$\eta\left(\{H \in \mathrm{Sub}(G) : h \in w_n H w_n^{-1}\}\right) \xrightarrow[n \to \infty]{} 1 \tag{5.4.1}$$

for every $\mathbf{w} = (w_n)_{n \geq 0} \in E \coloneqq \mathbf{bnd}^{-1}\left(\{H \in \mathrm{Sub}(G) : h \in H\}\right)$, where

$$\eta(E) = \mathbb{P}_\mu(\mathbf{bnd}^{-1}(\{H \in \mathrm{Sub}(G) : h \in H\})) = \eta(\{H \in \mathrm{Sub}(G) : h \in H\}) = c > 0.$$

Consider one such trajectory $(w_n)_{n\geq 0} \in E$. The set $\{w_n^{-1}hw_n : n \geq 0\}$ is almost surely infinite beacuse h has infinite conjugacy class.

Let $j \geq 2$. Since μ is non-degenerate, the action $G \curvearrowright (\mathrm{Sub}(G), \eta)$ is non-singular. Thus we can choose $\delta_j > 0$ such that for every measurable subset $A \subseteq \mathrm{Sub}(G)$ with $\eta(A) \geq 1-\delta_j$ we have

$$(g_i)_*\eta(A) \geq 1 - \frac{\varepsilon}{m2^j}$$

for every $i = 1, \ldots, m$. Using the convergence from Equation (5.4.1), we inductively construct an increasing sequence $(n_k)_{k\geq 2} \subseteq \mathbb{N}$ as follows. Let $n_2 \in \mathbb{N}$ be such that

$$\eta(\{H \in \mathrm{Sub}(G) : w_{n_2}^{-1}hw_{n_2} \in H\}) \geq 1 - \delta_2.$$

If $n_2, \ldots, n_k$ are already defined, find $n_{k+1} > n_k$ such that

$$w_{n_{k+1}}^{-1}hw_{n_{k+1}} \notin \left\{h_{w_{n_2}}, \ldots, h_{w_{n_k}}\right\} \text{ and } \eta(\{H \in \mathrm{Sub}(G) : w_{n_{k+1}}^{-1}hw_{n_{k+1}} \in H\}) \geq 1 - \delta_{k+1}.$$

Our choice of δ_k guarantees that

$$\eta(\{H : w_{n_k}^{-1}hw_{n_k} \in g_iHg_i^{-1}\}) = (g_i)_*\eta(\{H : w_{n_k}^{-1}hw_{n_k} \in H\}) \geq 1 - \frac{\varepsilon}{m2^k}$$

holds for every $k \geq 2$ and $i = 1, \ldots, m$. A union bound then shows that

$$\eta\left(\left\{H \in \mathrm{Sub}(G) : \; w_{n_k}^{-1}hw_{n_k} \notin \bigcap_{i=1}^{m} g_iHg_i^{-1}\right\}\right) \leq \frac{\varepsilon}{2^k}$$

for all $k \geq 2$, and hence that

$$\eta\left(\left\{H \in \mathrm{Sub}(G) : \; (w_{n_k}^{-1}hw_{n_k})_{k\geq 2} \not\subseteq \bigcap_{i=1}^{m} g_iHg_i^{-1}\right\}\right) \leq \varepsilon,$$

as desired, concluding the proof. □

5.4.2 Baumslag-Solitar groups and normalish subgroups

The Baumslag-Solitar groups are a family of one-relator groups

$$\mathrm{BS}(m,n) = \langle a, t \mid ta^mt^{-1} = a^n\rangle, \; m, n \in \mathbb{Z} \setminus \{0\}$$

introduced by Baumslag-Solitar in [BS62] that has been well-studied from geometric and algebraic perspectives [Why01, Lev07]. They are known to be C*-simple [dlHP11, Proposition 5] whenever m, n are both different from ± 1 and $m \neq \pm n$. The infinite cyclic subgroup $\langle a\rangle$ is always normalish since $\langle a\rangle/(\langle a\rangle \cap g\langle a\rangle g^{-1})$ is finite for all $g \in \mathrm{BS}(m,n)$ (see [Lev07, Section 2] for instance).

Proposition 5.4.2 below finds the parameters m, n such that $\mathrm{Sub}_{\mathrm{am}}(\mathrm{BS}(m,n))$ is countable. In particular, we deduce the following.

Corollary 5.4.1. *The group* $\mathrm{BS}(2,3)$ *is C*-simple, admits an amenable normalish subgroup and all its amenable μ-SRSs are trivial for every non-degenerate* $\mu \in \mathrm{Prob}(\mathrm{BS}(2,3))$.

Proof. The first two statements are already known from [dlHP11, Proposition 5] and [Lev07, Section 2], and Proposition 5.4.2 shows that BS(2, 3) has countably many amenable subgroups. Let μ be a non-degenerate probability measure on $G = \mathrm{BS}(2,3)$, and let η be an ergodic μ-SRS on $\mathrm{Sub}_{\mathrm{am}}(G)$. Then η has an atom, and by ergodicity must be supported in a finite orbit of some $H \in \mathrm{Sub}_{\mathrm{am}}(G)$. If H were non-trivial, then the normal subgroup generated by H would be a non-trivial normal amenable subgroup of G contradicting its C*-simplicity. Thus H is trivial and so is η. The case when η is non-ergodic follows from the ergodic decomposition. □

Proposition 5.4.2. *Let $m, n \in \mathbb{Z} \setminus \{0\}$ and $G = \mathrm{BS}(m,n)$. Then G has countably many amenable subgroups if and only if either*

- *n or m is equal to 1 or -1, or*
- *$n = m$ or $n = -m$, or*
- *m/n and n/m are not integers.*

The proof of one of the implications of Proposition 5.4.2 boils down to a result of Y. Cornulier [Cor] stating that $\mathrm{BS}(m,n)$ has no non-trivial element admiting infinitely many roots when m/n and n/m are not integers. Since this statement has only appeared in online forums we replicate its proof here for completeness.

For both implications, we will need to use the action of $\mathrm{BS}(m,n)$ on the *Bass-Serre tree $\mathcal{T}$ associated to its decomposition as an HNN-extension.* For our purposes, the tree $\mathcal{T}$ is the tree with vertex set $\mathrm{BS}(m,n)/\langle a\rangle$ and oriented edges connecting $w\langle a\rangle$ with $wa^l t\langle a\rangle$ for all $w \in \mathrm{BS}(m,n)$ and $0 \leq l \leq n-1$. Left multiplication induces an action of $\mathrm{BS}(m,n)$ on $\mathcal{T}$ through directed graph automorphisms. Bass-Serre theory ensures that $\mathcal{T}$ is indeed a tree and that the action of $\mathrm{BS}(m,n)$ is transitive on oriented edges and vertices. Thus the stabilizers of vertices (resp. edges) are conjugates of $\langle a\rangle$ (resp. $\langle a^m\rangle$). A non-trivial element of $\mathrm{BS}(m,n)$ stabilizing no vertex on $\mathcal{T}$ is said to be *loxodromic*, and is said to be *elliptic* otherwise. For more information on the general theory we refer to [Ser77, Chapitre I], and to [CGLMS25, Sections 1 & 2] for Baumslag-Solitar groups and their Bass-Serre trees.

We will need a lemma which is a small variation of [CGLMS25, Proposition 2.4].

Lemma 5.4.3. *If G is a countable group with a normal subgroup $N \subseteq G$ such that N has countably many amenable subgroups and every subgroup of G/N is finitely generated, then G has countably many subgroups.*

Proof. Write $\pi \colon G \to G/N$ the projection. A subgroup H of G is determined by $H \cap N$ and any section of a finite generating set of $\pi(H)$. There are always at most countably many options for these sections, and if H is amenable then there are at most countably many options for $H \cap N$. □

Proof of Proposition 5.4.2. We first prove the forward implication. If $G = \mathrm{BS}(\pm 1, m)$ then G has countably many amenable subgroups by [BLT19, Corollary 8.4]. If $n = \pm m$, then G is virtually

$F_n \times \mathbb{Z}$. The group $F_n \times \mathbb{Z}$ has countably many amenable subgroups by Lemma 5.4.3, and the same is true for G by Lemma 5.4.3 again.

Now suppose that m/n and n/m are not integers. Let $\mathcal{T}$ be the Bass-Serre tree associated to the HNN-extension decomposition of G as $G = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ and let $H \subseteq G$ be an amenable subgroup. Since H has no non-abelian free subgroups and the action of G on $\mathcal{T}$ has no inversions, by J. Tits' categorization of actions of groups on trees [Tit70] either H fixes a vertex, stabilizes the axis of a loxodromic element in H or fixes an end of $\partial\mathcal{T}$. In the first case H is cyclic. In the second case, the stabilizer of the axis ℓ of a loxodromic element in G splits as

$$1 \longrightarrow K \longrightarrow \mathrm{Stab}_G(\ell) \longrightarrow D_\infty \longrightarrow 1$$

where D_∞ is the infinite dihedral group and the kernel K is a subgroup of a vertex stabilizer (and hence either trivial or infinite cyclic). In any case, by Lemma 5.4.3 the countably many subgroups $\mathrm{Stab}_G(\ell)$ where ℓ runs through the axes of loxodromic elements of G have countably many subgroups.

We are reduced to the case when H fixes an end $\xi \in \partial\mathcal{T}$. Fix a one-sided geodesic $(\xi_n)_{n\geq 0}$ representing ξ, and define $K \subseteq G$ as the subgroup of elliptic elements fixing ξ. Then

$$K = \bigcup_{m\geq 0} \mathrm{Fix}_G(\xi_{[m,\infty)})$$

is the ascending union of the pointwise fixators of the rays $\xi_{[m,\infty)} = (\xi_{m+n})_{n\geq 0}$, $m \geq 0$. Thus K is either trivial, infinite cyclic or a direct union of infinite cyclic groups. In the last case, there is an element of $K \setminus \{e_G\}$ with infinitely many roots.

Claim ([Cor])**.** *If m/n and n/m are not integers, then $G = \mathrm{BS}(m,n)$ contains no non-trivial element with roots of arbitrarily large order.*

Proof of the claim. Let $h\colon G/\langle a\rangle \to \mathbb{Z}$ the *height function* associating to each coset $g\langle a\rangle$ the signed number of t's appearing in g. The function h is well defined (it factors through the quotient of G by the normal subgroup generated by a).

Let $g \in G \setminus \{e_G\}$ that fixes a vertex $v = w\langle a\rangle$ in $\mathcal{T}$, so $g = wa^{\zeta_g(v)}w^{-1}$ for a unique $\zeta_g(v) \in \mathbb{Z}\setminus\{0\}$. If g also fixes a vertex $v' = wa^l t\langle a\rangle$ in $\mathcal{T}$ that is adjacent to v (that is, there is an oriented edge from v to v' in $\mathcal{T}$), the equality

$$wa^{\zeta_g(v)}w^{-1} = wa^l ta^{\zeta_g(v')}(wa^l t)^{-1}$$

implies that $\zeta_g(v') = (m/n)\zeta_g(v)$. By induction we see that for any pair of vertices v, v' in the subtree $\mathcal{T}_g$ of fixed points of g in $\mathcal{T}$ we have $\zeta_g(v') = (m/n)^{h(v')-h(v)}\zeta_g(v)$. Since no non-trivial power of m/n is an integer and the $\zeta_g(v)$ are always integers, we deduce that h and *a fortiori* ζ_g are bounded on $\mathcal{T}_g$.

Now assume that $g \in G \setminus \{e_G\}$ has infinitely many roots, and let $M > 0$ be an upper bound for $\{\zeta_g(v) : v \in G/\langle a\rangle\}$. The action of g on $\mathcal{T}$ cannot be loxodromic, since the roots of g would have arbitrarily small translation length. Thus g and any of its roots are elliptic. Now let $u \in G$ with $u^N = g$ for some $N \in \mathbb{N}_+$, and let v' be a vertex of $\mathcal{T}$ fixed by u. Then

$$N \leq N|\zeta_u(v')| = |\zeta_g(v')| \leq M. \qquad \square$$

Thus K must be cyclic. If ξ is fixed by no loxodromic element of G then $H \subseteq \mathrm{Stab}_G(\xi) = K$, so there are countably many options for H.

If not, let $g \in G$ be a loxodromic element with minimal translation length $d \in \mathbb{N}_+$. Then $\mathrm{Stab}_G(\xi) = K \rtimes \mathbb{Z}$, where the generator of the right-hand side is g and conjugation by g sends every $\mathrm{Fix}_G(\xi_{[m,\infty)})$ to $\mathrm{Fix}_G(\xi_{[m+d,\infty)})$. Again, Lemma 5.4.3 shows that $\mathrm{Stab}_G(\xi)$ has countably many subgroups, and since there are countably many fixed ends in $\partial\mathcal{T}$ by loxodromic elements of G we conclude that H varies in a countable set. This finishes the proof that $\mathrm{BS}(m,n)$ has countably many amenable subgroups.

For the reverse implication, consider $m, n \in \mathbb{Z} \setminus \{0\}$ not satisfying any of the conditions in the statement of the proposition. Since $\mathrm{BS}(m,n)$ is isomorphic to $\mathrm{BS}(-m,-n)$ we may assume that $m > 0$, and since $\mathrm{BS}(m,n)$ contains $\mathrm{BS}(m^2,n^2)$ (see [Lev15, Theorem 1.3] for instance) we may assume that $n > 0$ too. Now $m \geq 2$ and we may write $n = km$ with $k \geq 2$.

As before, let $\mathcal{T}$ be the Bass-Serre tree associated to the HNN-extension decomposition of G. Let $\widetilde{\mathcal{T}}$ be the subtree of $\mathcal{T}$ with vertex set $\{wt^{-1}\langle a\rangle : w \text{ is a finite word in } \{a, t^{-1}\}\}$. The tree $\widetilde{\mathcal{T}}$ is a rooted complete m-ary tree with root $\langle a\rangle$.

Claim. *The fixed points of a^m in $\mathcal{T}$ are the vertices of $\widetilde{\mathcal{T}}$.*

Proof of the claim. The element a^m fixes $\langle a\rangle$ and does not fix any of its neighbors

$$t\langle a\rangle, at\langle a\rangle, a^2t\langle a\rangle, \cdots, a^{km-1}t\langle a\rangle$$

because $k \geq 2$, so the subtree $\mathcal{T}_{a^m}$ of fixed points of a^m is contained in $\widetilde{\mathcal{T}}$. Now let $wt^{-1}\langle a\rangle$ be a vertex of $\widetilde{\mathcal{T}}$, and write $w = a^{n_1}t^{-1}a^{n_2}t^{-1}\cdots a^{n_l}$ for some $l \in \mathbb{N}_+$ and $n_1, \dots, n_l \in \mathbb{N}$, so

$$\begin{aligned} a^m wt^{-1} &= a^m a^{n_1}t^{-1}a^{n_2}t^{-1}\cdots a^{n_l}t^{-1} = a^{n_1}t^{-1}a^{km}a^{n_2}t^{-1}\cdots a^{n_l}t^{-1} \\ &= a^{n_1}t^{-1}a^{n_2}t^{-1}a^{k^2m}\cdots a^{n_l}t^{-1} = \cdots = a^{n_1}t^{-1}a^{n_2}t^{-1}\cdots a^{n_l}t^{-1}a^{k^l m} \end{aligned}$$

and thus $a^m(wt^{-1}\langle a\rangle) = wt^{-1}\langle a\rangle$. We conclude that $\mathcal{T}_{a^m} = \widetilde{\mathcal{T}}$. □

Let ξ_1, ξ_2 be distinct ends of $\widetilde{\mathcal{T}}$ and identify them with geodesics in $\widetilde{\mathcal{T}}$ starting at $\langle a\rangle$. Let $gt^{-1}\langle a\rangle$ be a vertex of $\xi_1 \setminus \xi_2$, so $(gt^{-1})a^m(gt^{-1})^{-1}$ fixes ξ_1 and does not fix ξ_2. Thus the amenable subgroups $\mathrm{Stab}_G(\xi), \xi \in \partial\widetilde{\mathcal{T}}$ are pairwise distinct, and there are uncountably many subgroups of this type since $m \geq 2$. This finishes the proof of the proposition. □

5.4.3 Property (CS)

A countable group G is said to have *property (CS)* if, for every unitary representation $\pi\colon G \to B(\mathcal{H})$ that is weakly contained in the left-regular representation λ of G, there exists an neighborhood U of $\mathrm{id}_{\mathcal{H}}$ in the strong operator topology of $B(\mathcal{H})$ such that $\pi^{-1}(U)$ is contained in the amenable radical $R_a(G)$. Recall that π is *weakly contained in* λ if $\|\pi(f)\| \leq \|\lambda(f)\|$ for all $f \in \mathbb{C}[G]$. This notion was introduced in [BKKO17] as a sufficient condition for a group G to satisfy a conjecture of Connes-Sullivan on subgroups of connected Lie groups acting amenably on homogeneous spaces, proved by R. Zimmer [Zim78]. Linear groups and discrete groups G with $H^2_{\mathrm{b}}(G, \ell^2(G/R_a(G)) \neq 0$ are known to have property (CS) [BKKO17, Theorems 8.4 & 8.6].

Property (CS) is stronger than C^*-simplicity: every group with property (CS) and trivial amenable radical is C^*-simple [BKKO17, Proposition 8.2]. Indeed, a discrete group with no non-trivial finite normal subgroups and no amenable normalish subgroups is C^*-simple [BKKO17, Theorem 6.2], while property (CS) implies the absence of amenable normalish subgroups. To see this, it suffices to note that if a subgroup $H \subseteq G$ is amenable and normalish, then the quasi-regular representation $\lambda_{G/H}$ is weakly contained in the left regular representation, yet the image of G under $\lambda_{G/H}$ is not discrete in the strong operator topology. Combined with Theorem J, this yields the following.

Corollary 5.4.4. *Let G be a countable group with finite amenable radical and suppose that there exists a non-degenerate $\mu \in \mathrm{Prob}(G)$ which admits an amenable boundary μ-SRS distinct from a Dirac mass on a finite normal subgroup of G. Then G does not have property (CS).*

In fact, the existence of an amenable boundary μ-SRS η allows us to explicitly construct a unitary representation of G that witnesses the failure of property (CS). Following the notation of [BdlH20, Section F.5], let us define the direct integral $\mathcal{H}_\Pi$ of $(\ell^2(G/H))_{H \in \mathrm{Sub}_{\mathrm{am}}(G)}$ as the Hilbert space

$$\mathcal{H}_\Pi = \int_{\mathrm{Sub}_{\mathrm{am}}(G)} \ell^2(G/H)\,\mathrm{d}\eta(H),$$

that is, $\mathcal{H}_\Pi$ consists of all maps $H \in \mathrm{Sub}_{\mathrm{am}}(G) \mapsto v_H \in \ell^2(G/H)$ such that

$$\int_{\mathrm{Sub}_{\mathrm{am}}(G)} \|v_H\|^2_{\ell^2(G/H)}\,\mathrm{d}\eta(H) < \infty$$

and such that $H \in \mathrm{Sub}_{\mathrm{am}}(G) \mapsto (v_H)_{gH}$ is measurable for every $g \in G$. Then, $\mathcal{H}_\Pi$ is a Hilbert space with inner product

$$\langle v, w\rangle = \int_{\mathrm{Sub}_{\mathrm{am}}(G)} \langle v_H, w_H\rangle_{\ell^2(G/H)}\,\mathrm{d}\eta(H)$$

for every $v = (v_H)_H$, $w = (w_H)_H \in \mathcal{H}_\Pi$.

Definition 5.4.5. *We define the direct integral of the family of all quasi-regular representations $(\lambda_{G/H})_{H \in \mathrm{Sub}_{\mathrm{am}}(G)}$ as the unitary representation $\Pi\colon G \to B(\mathcal{H}_\Pi)$ defined by $(\Pi(g)v)_H = \lambda_{G/H}(g)v_H$ for every $g \in G$, $v \in \mathcal{H}_\Pi$ and $H \in \mathrm{Sub}_{\mathrm{am}}(G)$. Define π to be the cyclic subrepresentation of Π generated by the vector $v = (\delta_H)_{H \in \mathrm{Sub}_{\mathrm{am}}(G)} \in \mathcal{H}_\Pi$.*

Every quasi-regular representation $\lambda_{G/H}$ is weakly contained in the left-regular representation λ, since $H \in \mathrm{Sub}_{\mathrm{am}}(G)$. Therefore, π is weakly contained in λ. The next result shows that such representation provides a concrete witness showing that the group G cannot satisfy property (CS).

Proposition 5.4.6. *Let G be a countable group with finite amenable radical and suppose that there exists a non-degenerate $\mu \in \mathrm{Prob}(G)$ which admits an amenable boundary μ-SRS distinct from a Dirac mass on a finite normal subgroup of G.*

Let π be the cyclic subrepresentation of the direct integral representation Π generated by the vector $v = (\delta_H)_{H\in \mathrm{Sub}_{\mathrm{am}}(G)}$ (Definition 5.4.5). Then, the identity operator in $B(\mathcal{H}_\pi)$ is not isolated in the strong operator topology.

Proof. We show that, for every SOT-neighborhood U of the identity in $B(\mathcal{H}_\pi)$, there exists an element $g \in G\setminus\{e_G\}$ such that $\pi(g) \in U$. Since the representation π is generated by v, it suffices to verify this for neighborhoods of the form

$$U = \{T \in B(\mathcal{H}_\pi) : \left\|\pi(g_i^{-1})T\pi(g_i)v - v\right\|_{\mathcal{H}_\pi} < \varepsilon \text{ for all } i = 1, \dots, m\}$$

for every $\varepsilon > 0$ and $g_1, \dots, g_m \in G$.

Now,one can follow the proof of Theorem J for $j = 2$. Indeed, the convergence from Equation (5.4.1) and the non-singularity of the action $G \curvearrowright \mathrm{Sub}_{\mathrm{am}}(G)$ together imply that there exists $k \in \mathbb{N}$ and $(w_n)_{n\geq 0} \in G^{\mathbb{N}}$ such that $\eta(\{H \in \mathrm{Sub}_{\mathrm{am}}(G) : w_k h w_k^{-1} \in H\}) \geq 1 - \delta$, where $\delta > 0$ is chosen in such a way that

$$(g_i)_*\eta(\{H \in \mathrm{Sub}_{\mathrm{am}}(G) : w_k^{-1} h w_k \in H\}) = \eta(\{H \in \mathrm{Sub}_{\mathrm{am}}(G) : w_k^{-1} h w_k \in g_i H g_i^{-1}\})$$

is at most $1 - \varepsilon^2/(2m)$ for every $i = 1, \dots, m$. Set $g := w_k^{-1} h w_k \in G\setminus\{e_G\}$. A union bound then shows that

$$\eta(\{H \in \mathrm{Sub}_{\mathrm{am}}(G) : g_i^{-1} g g_i \in H\}) > 1 - \varepsilon^2/2.$$

We obtain

$$\begin{aligned}\left\|\pi(g_i^{-1} g g_i)v - v\right\|_{\mathcal{H}_\pi}^2 &= \int_{\mathrm{Sub}_{\mathrm{am}}(G)} \left\|\lambda_{G/H}(g_i^{-1} g g_i)\delta_H - \delta_H\right\|_{\mathcal{H}_\pi}^2 \mathrm{d}\eta(H) \\ &= 2\eta(\{H : g_i^{-1} g g_i \notin H\}) \leq \varepsilon^2\end{aligned}$$

for every $i = 1, \dots, m$, which finally shows that $\pi(g) \in U$. $\square$

Chapter 6

Groups with classifiable actions on the line

This chapter corresponds to the preprint [BGVMB26], **and represents joint work with Joaquín Brum (Universidad de la República) and Nicolás Matte Bon (CNRS & Institut Camille Jordan)**.

We motivate and study the class $\mathcal{C}$ of countable groups G such that the conjugacy relation between minimal actions of G on $\mathbb{R}$ by orientation-preserving homeomorphisms has a Borel transversal. We show a number of stability properties of $\mathcal{C}$ under group-theoretic operations and that $\mathcal{C}$ contains all finitely generated groups of piecewise projective homeomorphisms of the line. We also prove that the semiconjugacy relation among cocompact actions of a countable group G is essentially countable, and that is is smooth if and only if $G \in \mathcal{C}$.

6.1 Introduction

Context

One of the ultimate goals of the study of actions of a countable group G on a one-manifold X by orientation-preserving homeomorphisms is the classification, in a suitable sense, of all such actions of G. The situation when X is the circle is somewhat well-behaved, at least when no considerations for the regularity of such homeomorphisms are involved. Indeed, a generalization of a theorem of Poincaré for circle homeomorphisms states that an action $\varphi\colon G \to \mathrm{Homeo}_0(S^1)$ always admits a *minimal set*, that is, a non-empty closed φ-invariant subset Λ of S^1 such that action of $\varphi(G)$ on Λ has only dense orbits. When φ has no finite orbits, this allows one to reduce the study of φ to the case when $\Lambda = S^1$. We are thus led to consider the Polish space $\mathrm{Rep}_{\min}(G, S^1)$ of minimal actions of G on S^1 and the action of the group $\mathrm{Homeo}_0(S^1)$ of orientation-preserving homeomorphisms of S^1 on $\mathrm{Rep}_{\min}(G, S^1)$ by conjugation.

A theorem of É. Ghys [Ghy87] shows that two minimal actions on the circle are conjugate if and only if they have the same bounded Euler class. S. Matsumoto [Mat86] gives a more elementary interpretation of this result. The existence of an explicit invariant that classifies a class of actions up to conjugacy — regardless of the invariant — is a result in its own right that can be formalized in the framework of Borel equivalence relations. It is apparent from [Mat86] that there is a Borel map from $\mathrm{Rep}_{\min}(G, S^1)$ to a standard Borel space such that its fibers are exactly the conjugacy classes. The conjugacy relation on $\mathrm{Rep}_{\min}(G, S^1)$ is thus said to be *smooth*.

The situation when X is the real line is more complicated and is the subject of this chapter. Firstly, even when an action by orientation-preserving homeomorphisms of $\mathbb{R}$ has no global fixed points it may fail to admit a minimal set. This phenomenon cannot happen when G is finitely generated, but, secondly, even in this case there may be no Borel map from the space $\mathrm{Rep}_{\min}(G, \mathbb{R})$ of minimal actions of G on $\mathbb{R}$ to a standard Borel space whose fibers are the conjugacy classes. For instance, the free group on two generators admits no such map (see Subsection 6.2.2 below). On the other hand, a foundational result of O. Hölder [Höl01] shows that all minimal actions of countable abelian groups on the real line are conjugate to actions by a dense group of translations, and this remains true for groups containing no non-abelian free semigroups due to work of J. Plante [Pla75]. For another example see [Riv10], where C. Rivas proves that every solvable Baumslag-Solitar group $\mathrm{BS}(1, n), n \in \mathbb{N}_+$ admits only two minimal actions on $\mathbb{R}$ up to conjugacy.

Main results

We will concentrate on the second issue brought up in the previous paragraph, and specifically on the class of groups where a Borel classification of its minimal actions on the real line is possible. For a countable group G, we denote by $\mathrm{Rep}_{\min}(G)$ the space of minimal actions of G on $\mathbb{R}$ by orientation-preserving homeomorphisms.

Definition 6.1.1. *Define $\mathcal{C}$ as the class of countable groups G such that the conjugacy relation on $\mathrm{Rep}_{\min}(G)$ is smooth.*

Given the preceding examples, one may expect the answer to the following questions to be positive.

Question 6.1.2. Is every amenable group contained in $\mathcal{C}$? Is every elementary amenable group contained in $\mathcal{C}$?

At the present time we do not know the answer. Our main results show that $\mathcal{C}$ is a large class in any case. Recall that a subgroup $H \leq G$ is *commensurated* if $gHg^{-1} \cap H$ has finite index in H for every $g \in G$. We denote by $\langle\langle H \rangle\rangle$ its normal closure in G.

Theorem L. *Let G be a finitely generated group. If there is a commensurated subgroup $H \leq G$ such that H and $G/\langle\langle H \rangle\rangle$ belong to $\mathcal{C}$, then $G \in \mathcal{C}$.*

Corollary 6.1.3.

i. If G is a finitely generated group having a normal subgroup $N \trianglelefteq G$ such that N and G/N belong to $\mathcal{C}$, then $G \in \mathcal{C}$.

ii. If $H, K \in \mathcal{C}$ are finitely generated, then the wreath product $H \wr K$ is in $\mathcal{C}$.

iii. If $\Phi \colon H \to K$ is an isomorphism between finite-index subgroups H, K of a finitely generated group $G \in \mathcal{C}$, then the HNN-extension G^{Φ} is in $\mathcal{C}$.

iv. If $H, G \in \mathcal{C}$ are finitely generated groups containing K as a finite-index subgroup, then the amalgamated product $G *_K H$ is in $\mathcal{C}$.

Remarks.

- From item (i) it follows by induction that if G is a finitely generated group having a series of normal subgroups
$$\{e\} = G_0 \trianglelefteq \cdots \trianglelefteq G_m = G$$
such that $G_i/G_{i-1} \in \mathcal{C}$ for $i = 1, \ldots, m$ (for instance, if G_i/G_{i-1} does not contain a free semigroup on two generators), then $G \in \mathcal{C}$. In particular, $\mathcal{C}$ contains all finitely generated virtually solvable groups: this could also be deduced from the more detailed results on actions of solvable groups on the line in [BMBRT25, Theorem B].

- In view of Corollary 6.1.3 (i), an affirmative answer to Question 6.1.2 for elementary amenable groups would follow if $\mathcal{C}$ were also stable under direct limits. Unfortunately this is not true: a counterexample is constructed in the proof of Theorem 6.1.4 (which will not appear in this thesis), but is not elementary amenable. It is nevertheless true that an infinite *direct sum* of finitely generated groups in $\mathcal{C}$ is still in $\mathcal{C}$ (Proposition 6.5.4), which gives the stability under wreath product in item (ii). This is a rich source of examples in view of the fact that the wreath product $H \wr K$ of any two countable groups acting faithfully on $\mathbb{R}$ always has *minimal* faithful actions on $\mathbb{R}$ (see [BMBRT24, Example 8.1.8]).

- Items (iii) and (iv) in Corollary 6.1.3 are both special cases of a statement about fundamental groups of graphs of groups (Corollary 6.5.7), which in turn follows from Theorem L. These imply that all Baumslag-Solitar groups $\mathrm{BS}(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ for $m, n \in \mathbb{Z} \setminus \{0\}$ belong to $\mathcal{C}$. When $n, m \geq 1$, these groups are known to admit faithful minimal actions on $\mathbb{R}$ by a construction of Farb and Franks [FF20].

From [BMBRT24, Section 16.3.1] it was already known that Thompson's group F of piecewise dyadically affine homeomorphisms of the interval belongs to $\mathcal{C}$. We extend this result to any finitely generated group G which can be embedded in the group $\mathrm{PProj}_0(\mathbb{R})$ of orientation-preserving piecewise projective homeomorphisms of the line, with no assumptions on the embedding. Here, a homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{R})$ belongs to $\mathrm{PProj}_0(\mathbb{R})$ if, outside a finite subset of $\mathbb{R}$, it is locally of the form $x \mapsto (ax + b)/(cx + d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Theorem M. *Let $G \subseteq \mathrm{PProj}_0(\mathbb{R})$ be finitely generated. Then $G \in \mathcal{C}$. In particular, any finitely generated group of piecewise affine homeomorphisms of $\mathbb{R}$ belongs to $\mathcal{C}$.*

The amenability of groups of piecewise affine homeomorphisms of $\mathbb{R}$ is an outstanding open problem, the most well-known special case being the amenability of Thompson's group F. Theorem M shows that, when finitely generated, these groups are all in $\mathcal{C}$, independent of the answer to this problem. This class is also a rich source of (non-virtually solvable) elementary amenable groups, by work of M. Brin [Bri05] (see also [Nav04]) and of Bleak-Brin-Moore [BBM21]. This might appear as encouraging evidence towards an affirmative answer to Question 6.1.2. However, finitely generated groups of piecewise projective homeomorphisms can be non-amenable [Mon13] (see [LM16] for an explicit example) thus showing that Theorem M does not use amenability.

Question 6.1.2 for amenable groups may be extremely hard to decide: a negative answer seems to require fundamentally new ways to construct amenable groups acting on the line. The following theorem shows that an affirmative answer would imply that Thompson's F is not amenable.

Theorem 6.1.4. *There is a finitely generated group $G \notin \mathcal{C}$, such that G is amenable if and only if Thompson's group F is amenable.*

The proof of the previous theorem appears in the preprint [BGVMB26], but does is not presented in this thesis since its proof predates the author's involvement in the project.

6.1.1 Beyond the class $\mathcal{C}$: the Borel complexity of conjugacy

When leaving the class $\mathcal{C}$, one can still try to rank groups according to the richness of their space of minimal actions on the line. This problem can be framed within the theory of Borel reducibility of equivalence relations, which allows to compare different classification problems in mathematics formulated as Borel or analytic equivalence relations on standard Borel spaces. We refer to [Hjo00, BK96] for a presentation of the subject.

Given equivalence relations E_1, E_2 defined on standard Borel spaces Z_1, Z_2 respectively, we say that E_1 *is reducible to* E_2 if there exists a Borel map $r\colon Z_1 \to Z_2$ such that for all $x, y \in Z_1$, we have $(x, y) \in E_1$ if and only if $(r(x), r(y)) \in E_2$. This notion expresses in a rigorous way the idea that deciding if elements of Z_2 are E_2-related is at least as hard as deciding if elements of Z_1 are E_1-related. We say two relations are *bireducible* if one is reducible to the other and vice-versa. A Borel equivalence relation is *smooth* if it is reducible to the identity on some standard Borel space, so such relations are the simplest ones from this point of view.

The largest natural space of actions of a countable group G on $\mathbb{R}$ that one may expect to understand is the space $\mathrm{Rep}_{\mathrm{irr}}(G)$ of actions with no global fixed point on $\mathbb{R}$, called *irreducible* actions. A classical construction by A. Denjoy [Den32] by blowing up orbits of an irreducible action φ allows to produce many actions with roughly the same dynamics as φ but not conjugate to it. The equivalence relation generated by the blow-up procedure coincides with semiconjugacy: two irreducible actions φ_1, φ_2 are *semiconjugate* if there exists a non-decreasing (and not necessarily continuous) map $h\colon \mathbb{R} \to \mathbb{R}$ that intertwines φ_1 with φ_2. Semiconjugacy among irreducible actions is a natural equivalence relation in this context, and it boils down to conjugacy when both actions are minimal.

Write $\mathbb{R} \curvearrowright^{\Psi} \mathrm{Rep}_{\mathrm{irr}}(G)$ for the *translation flow* conjugating an irreducible action by translations. Following [DKNP13, Der13a], if G is finitely generated the study of symmetric random walks on G allows one to construct a compact Ψ-invariant subspace $\mathrm{Harm}(G) \subseteq \mathrm{Rep}_{\mathrm{irr}}(G)$, the *space of harmonic actions* of G, composed of minimal actions and actions of G by translations together with a continuous retraction map $r\colon \mathrm{Rep}_{\mathrm{irr}}(G) \to \mathrm{Harm}(G)$ that reduces semiconjugacy on $\mathrm{Rep}_{\mathrm{irr}}(G)$ to the orbit equivalence relation of Ψ on $\mathrm{Harm}(G)$. It was noticed in [BMBRT24, Section 14.4] that the existence of these objects, along with a theorem of V. Wagh [Wag88] shows the following.

Proposition 6.1.5 ([BMBRT24, Section 14.4])**.** *Let G be a finitely generated group. The semiconjugacy relation between irreducible actions of G on the line is essentially hyperfinite, and it is smooth if and only if $G \in \mathcal{C}$.*

Here, a relation is *essentially hyperfinite* if it is bireducible to the orbit equivalence relation of a Borel automorphism on a standard Borel space. These form a strictly larger class of relations than the smooth ones, albeit one of low complexity (see Subsection 6.2.2 below).

An irreducible action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ is said to be *cocompact* if there is a compact subset of $\mathbb{R}$ intersecting all $\mathbb{R}$-orbits. Equivalently, the action φ admits a minimal set (see [Nav11, Proposition 2.1.2]). When G is not finitely generated, we do not know if a suitable harmonic space exists for G which would control all cocompact actions of G in a similar way as in the finitely generated case. We are able to show nonetheless that semiconjugacy among cocompact actions is *essentially countable*, that is, bireducible to a Borel equivalence relation where every equivalence class is countable. In particular, it is a Borel equivalence relation, which is not evident *a priori*.

Theorem N. *Let G be a countable group. The semiconjugacy relation between cocompact actions of G on the line is essentially countable, and it is smooth if and only if $G \in \mathcal{C}$.*

Question 6.1.6. Let G be a countable group. Is the semiconjugacy relation between cocompact actions of G on the line always essentially hyperfinite?

To the best of our knowledge these questions are unrelated to the study of the Borel complexity of a countable group acting on its space of left orders, which has been undertaken in [CC22, CC24].

6.1.2 On proofs

The main criterion we use to verify membership in $\mathcal{C}$ is Theorem 6.3.4, which asserts that a group G is in $\mathcal{C}$ if and only if for every sequence $(f_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$ and every proximal minimal action φ of G such that $\lim_{n\to\infty} f_n.\varphi = \varphi$ we have $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$ (that is, any sequence of homeomorphisms almost centralizing φ is asymptotically trivial). Equipped with this criterion, the proof of Theorem L relies on the structural theory of group actions on the line and the notion of laminar actions introduced in [BMBRT24, Chapter 8] in a fundamental way.

The proof of Theorem M also involves structure theory for actions of micro-supported groups from [BMBRT24, Chapter 9].

Even though our main results deal with finitely generated groups, we do not formulate the definition of the class $\mathcal{C}$ in terms of harmonic spaces $\mathrm{Harm}(G)$. We do so since our definition of $\mathcal{C}$ is natural, and to prove closure properties of $\mathcal{C}$ it is crucial to consider groups that are not finitely generated.

6.1.3 Organization of the chapter

Section 6.2 collects the necessary preliminaries to read this chapter. In Section 6.3 we prove Theorem 6.3.4, which is the main criterion we use to decide membership of a group in $\mathcal{C}$. Section 6.4 records some restrictions on groups acting minimally coming from commensurated subgroups, to be used in the proof of Theorem L. Sections 6.5 proves Theorem L and Corollary 6.1.3. Sections 6.6 and 6.7 prove Theorems M and N respectively.

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6.2 Preliminaries

We review the theory of group actions on the line and some basic vocabulary on Borel equivalence relations. For more details on this material, see [DNR16, Nav11] and [BK96]. In this section G will always denote a countable group.

Conventions

A *Radon measure on* $\mathbb{R}$ is a non-zero regular Borel measure on $\mathbb{R}$, and thus gives finite mass to compact sets. We write $\mathrm{Homeo}_0(\mathbb{R})$ for the group of orientation-preserving homeomorphisms of the real line. Given an action $\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R})$ and a subgroup $H \subseteq G$ we write $\mathrm{Fix}_\varphi(H)$ for the closed set of $x \in \mathbb{R}$ such that $\varphi(h).x = x$ for all $h \in H$, and $\mathrm{supp}_\varphi(H) = \mathbb{R} \setminus \mathrm{Fix}_\varphi(H)$. The *centralizer* of $\varphi(H)$ is the group $\mathrm{Cent}_\varphi(H) \subseteq \mathrm{Homeo}_0(\mathbb{R})$ of homeomorphisms commuting with all elements in $\varphi(H)$.

If $K, H \subseteq G$ are subgroups, we say that K and H are *commensurate* if $K \cap H$ has finite index in K and in H, and we say that H is *commensurated in* G if for all $g \in G$, $H \cap gHg^{-1}$ has finite index in H and in gHg^{-1}. For us, conjugation by some $a \in G$ is the map $g \in G \mapsto aga^{-1}$. We denote by $\langle\langle H\rangle\rangle$ the normal subgroup of G generated by $H \subseteq G$.

6.2.1 Actions on the line

An action $\varphi \in \mathrm{Hom}(G, \mathrm{Homeo}_0(\mathbb{R}))$ is said to be *irreducible* if it has no global fixed points on $\mathbb{R}$, and we denote the set of irreducible actions of G by $\mathrm{Rep}_{\mathrm{irr}}(G)$. Two actions $\varphi_1, \varphi_2 \in \mathrm{Rep}_{\mathrm{irr}}(G)$ are *semiconjugate* if there is a non-decreasing map $h\colon \mathbb{R} \to \mathbb{R}$, called a *semiconjugacy*, such that $h \circ \varphi_1(g) = \varphi_2(g) \circ h$ for every $g \in G$. Semiconjugacy is an equivalence relation on $\mathrm{Rep}_{\mathrm{irr}}(G)$. Notice that the map h is not necessarily continuous, however it is automatically a homeomorphism when both actions are minimal. When the map h is continuous, we shall refer to it as a *continuous semiconjugacy from* φ_1 *to* φ_2 (this defines a transitive but not symmetric relation).

We say that a closed non-empty subset $\Lambda \subseteq \mathbb{R}$ is a *minimal set* for an action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ if Λ is φ-invariant and every φ-orbit in Λ is dense in Λ. The set Λ may be either

- a discrete orbit, in which case φ is semiconjugate to an action that factors through $\mathbb{Z}$, or
- all of $\mathbb{R}$, or
- a perfect, totally disconnected and unbounded subset of $\mathbb{R}$, in which case by collapsing the intervals in $\mathbb{R} \setminus \Lambda$ we obtain a continuous semiconjugacy from φ to a minimal action of G on $\mathbb{R}$.

When Λ is not a discrete orbit it is the unique minimal set, and if G is finitely generated φ always admits a minimal set (see [DNR16, Lemma 3.5.18]). In general, the existence of a minimal set for φ is equivalent to φ being *cocompact*, that is, such that there is a compact subset of $\mathbb{R}$ intersecting every φ-orbit (see [Nav11, Proposition 2.1.2]). In this case, any closed φ-invariant subset of $\mathbb{R}$ contains a minimal set.

Given $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$, two points $x, y \in \mathbb{R}$ are said to be *proximal for* φ if there is $z \in \mathbb{R}$ and a sequence $(g_n)_{n\geq 0} \subseteq G$ such that $\lim_{n\to\infty} \varphi(g_n).x = \lim_{n\to\infty} \varphi(g_n).y = z$. The action φ is said to be *proximal* if every pair of points of $\mathbb{R}$ is proximal for φ, and is said to be *locally proximal* if every point in $\mathbb{R}$ is contained in an open interval whose endpoints are proximal for φ. When φ admits a minimal set a further trichotomy is true.

Theorem 6.2.1 ([Mar00, Ant84])**.** *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ and suppose that φ admits a minimal set. Then either:*

(I) the action is semiconjugate to an action by translations, or

(II) the action is locally proximal and not proximal, or

(III) the action is proximal.

If φ is proximal, then G contains a non-abelian free semigroup. If φ is locally proximal and not proximal, then G contains a non-abelian free group.

We say that φ is of *type I*, *type II* or *type III* according to which of the previous alternatives is verified. When φ is minimal, these three cases are distinguished by their centralizer $\mathrm{Cent}_\varphi(G)$:

the action φ is of type I if $\mathrm{Cent}_\varphi(G)$ is conjugate to the group of translations, of type II if $\mathrm{Cent}_\varphi(G)$ is infinite cyclic, and of type III if $\mathrm{Cent}_\varphi(G)$ is trivial. Type I actions further decompose into those semiconjugate to a group of translations acting minimally on $\mathbb{R}$, and those that are semiconjugate to a *cyclic* action, that is, an action factoring through a morphism $G \to \mathbb{Z}$.

6.2.2 Borel equivalence relations

A *standard Borel space* is a measurable space isomorphic to the Borel σ-algebra of a *Polish space*, that is, a complete separable metric space. An equivalence relation E on a standard Borel space Z is said to be *Borel* if $E \subseteq Z \times Z$ is Borel, and is said to be *countable* if all equivalence classes of E are countable. We denote the E-equivalence class of an element $z \in Z$ by $[z]_E$. Recall that, given equivalence relations E_1, E_2 defined on standard Borel spaces Z_1, Z_2 respectively we say that E_1 *is reducible to* E_2 if there exists a Borel map $r\colon Z_1 \to Z_2$ such that for all $x, y \in Z_1$, we have $(x, y) \in E_1$ if and only if $(r(x), r(y)) \in E_2$, and that E_1 is *bireducible* to E_2 if E_1 is reducible to E_2 and viceversa.

Studying the bireducibility class of the conjugacy relation for different classes of topological or measurable group actions is a question that has already spurred much research, see the monographs [Hjo00, BK96]. In our context we are only interested in the following two bireducibility types. As was stated in Section 6.1, a Borel equivalence relation E on Z is *smooth* if there exists a Borel map $Z \to \mathbb{R}$ whose fibers are the E-classes. This condition is invariant under bireducibility, and it is equivalent to saying that the quotient measurable space Z/E is standard. A larger class of Borel equivalence relations are the *essentially hyperfinite* ones, which can be defined as the ones bireducible to a countable Borel equivalence relation $\widetilde{E}$ that is the orbit equivalence relation of a Borel action of the integers $\mathbb{Z}$ on a standard Borel space Z [SS88].

A distinguished example of a non-smooth hyperfinite equivalence relation is $\mathcal{E}_0$, the relation on $\{0,1\}^{\mathbb{N}}$ where $((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) \in \mathcal{E}_0$ if and only if there is an $m \in \mathbb{N}$ with $x_n = y_n$ for all $n \geq m$. A result of Dougherty-Jackson-Kechris [DJK94] states that $\mathcal{E}_0$ is the unique non-smooth and essentially hyperfinite Borel equivalence relation up to bireducibility. Moreover, $\mathcal{E}_0$ is reducible to *any* non-smooth (and not necessarily essentially hyperfinite) Borel equivalence relation [HKL90]. We will make use of this theorem, called the *Glimm-Effros dichotomy*, in the special case when the Borel equivalence relation is F_σ (that is, a countable union of closed sets) and is generated by the action of a Polish group on a Polish space.

Theorem 6.2.2 ([Eff81], see [BK96, Theorem 3.4.2])**.** *Let Γ be a Polish group and Y a Polish space equipped with a continuous action of Γ. Let $E \subseteq Y \times Y$ be the orbit equivalence relation of the action, and suppose that E is F_σ. Then either E is smooth or $\mathcal{E}_0$ is reducible to E.*

Moreover, the first alternative holds if and only if for every $y \in Y$, the map

$$\gamma \mathrm{Stab}_\Gamma(y) \in \Gamma/\mathrm{Stab}_\Gamma(y) \mapsto \gamma.y \in \mathrm{Orb}_\Gamma(y)$$

is a homeomorphism.

Example 6.2.3. We restate an example from [BMBRT24, Section 14.4] showing that $\mathcal{E}_0$ Borel reduces to conjugacy among minimal (even faithful) actions of the free group F_2 on the line.

Consider $\widetilde{g}, \widetilde{h}$ two homeomorphisms of $\mathbb{R}/\mathbb{Z}$ with $\mathrm{Fix}(\widetilde{g}) = \{0, 1/2\}$, $\mathrm{Fix}(\widetilde{h}) = \{1/4, 3/4\}$ such that $\langle \widetilde{g}, \widetilde{h} \rangle$ acts minimally on $\mathbb{R}/\mathbb{Z}$. Let g, h be lifts of $\widetilde{g}, \widetilde{h}$ to the line with fixed points $\mathrm{Fix}(g) = \frac{1}{2}\mathbb{Z}$ and $\mathrm{Fix}(h) = 1/4 + \frac{1}{2}\mathbb{Z}$. Given a word $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$ define a homeomorphism g_ω by $g_\omega(x) = g^{\omega_n}(x)$ for every $x \in [n/2, (n+1)/2]$, $n \in \mathbb{Z}$, and let φ_ω be the action of F_2 defined by g_ω, h. Then the orbits of φ_ω are the same of those of $\langle g, h \rangle$, so φ_ω is minimal. Two such actions φ_ω, $\varphi_{\omega'}$ are conjugate if and only if ω, ω' belong to the same orbit of the bilateral shift: indeed, any conjugacy between φ_ω, $\varphi_{\omega'}$ preserves $\frac{1}{2}\mathbb{Z}$, and for every $n \in \mathbb{Z}$ the condition $\omega_n = 1$ can be recognized from the sign of $g_\omega - \mathrm{id}_{\mathbb{R}}$ on $[n/2, (n+1)/2]$.

We will also need the following strong uniformization theorem. Recall that a K_σ set in a Polish space is a countable union of compact sets.

Theorem 6.2.4 ([Ars40, Kun40], see [Kec95, (18.18)]). *Let Z be a standard Borel space, Y a Polish space and $B \subseteq Y \times Z$ a Borel subset. Denote by $\pi\colon Y \times Z \to Y$ the projection onto the first coordinate. Suppose that the sections $\pi^{-1}(y) \cap B$ are K_σ for every $y \in Y$. Then $\pi(B)$ is a Borel subset, and there exists a Borel function $\zeta\colon \pi(B) \to B$ such that $\pi \circ \zeta = \mathrm{id}_{\pi(B)}$.*

6.2.3 The space of harmonic actions on the line

Let G be a finitely generated group. Equip $\mathrm{Rep}_{\mathrm{irr}}(G)$ with the topology coming from the inclusion $\mathrm{Rep}_{\mathrm{irr}}(G) \subseteq \mathrm{Homeo}_0(\mathbb{R})^G$ and define the *translation flow* $\mathbb{R} \curvearrowright^{\Psi} \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ by

$$\Psi^t(\varphi)(g) = T_t \circ \varphi(g) \circ T_{-t}$$

for all $g \in G$ and $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$, where $T_t\colon s \mapsto s + t$ is the translation. Notice that Ψ defines a continuous action of $\mathbb{R}$.

We say that two actions $\varphi_1, \varphi_2 \in \mathrm{Rep}_{\mathrm{irr}}(G)$ are *pointed semiconjugate* if there is an action $\eta \in \mathrm{Rep}_{\mathrm{irr}}(G)$ that is minimal or cyclic, and semiconjugacies $h_i\colon \mathbb{R} \to \mathbb{R}$ between φ_i and η for $i = 1, 2$ such that $h_1(0) = h_2(0)$.

Theorem 6.2.5 ([DKNP13, Theorem 8.5]). *Let G be a finitely generated group. Then there exists a compact Ψ-invariant subspace $\mathrm{Harm}(G) \subseteq \mathrm{Rep}_{\mathrm{irr}}(G)$ composed of minimal and cyclic actions, and a continuous retraction $r\colon \mathrm{Rep}_{\mathrm{irr}}(G) \to \mathrm{Harm}(G)$ such that two actions $\varphi_1, \varphi_2 \in \mathrm{Rep}_{\mathrm{irr}}(G)$ are pointed semiconjugate if and only if $r(\varphi_1) = r(\varphi_2)$. In particular, φ_1, φ_2 are semiconjugate if and only if $r(\varphi_1), r(\varphi_2)$ are in the same Ψ-orbit.*

It is shown in [BMBRT24, Section 14.2] that the map r can be chosen to be continuous, and hence is a Borel reduction of semiconjugacy to the orbit equivalence relation of a Borel flow. Such a relation is essentially hyperfinite [Wag88], and as a consequence the previous theorem exhibits a dichotomy for finitely generated groups acting on the line: either their semiconjugacy relation is smooth or it is bireducible to $\mathcal{E}_0$.

In this context, Theorem 6.2.2 already gives a criterion for finitely generated groups to belong to $\mathcal{C}$ since the orbit equivalence relations of an $\mathbb{R}$-flow is always F_σ. Here, a *recurrent point* of Ψ is an action $\varphi \in \mathrm{Harm}(G)$ such that for every open $U \subseteq \mathrm{Harm}(G)$ the set of return times $\{t \in \mathbb{R} : \Psi^t(\varphi) \in U\}$ is unbounded.

Proposition 6.2.6 ([BMBRT24, Corollary 14.4.2]). *A finitely generated group G belongs to $\mathcal{C}$ if and only if every Ψ-recurrent element of* $\mathrm{Harm}(G)$ *is Ψ-periodic. If this is not the case, then the semiconjugacy relation on* $\mathrm{Rep}_{\mathrm{irr}}(G)$ *(equivalently, the conjugacy relation on the space of minimal actions of G) is bi-reducible to $\mathcal{E}_0$.*

Remark 6.2.7. Notice that for any $g \in G$ and $\varphi \in \mathrm{Harm}(G)$ the constant

$$\sup\{|\varphi(g).x - x| : x \in \mathbb{R}\}$$

is finite. Indeed, for every $g \in G$, by Ψ-invariance of $\mathrm{Harm}(G)$ we have

$$\begin{aligned}\sup\{|\varphi(g).x - x| : \varphi \in \mathrm{Harm}(G),\, x \in \mathbb{R}\} &= \sup\{\left|\Psi^t(\varphi)(g).0\right| : \varphi \in \mathrm{Harm}(G),\, t \in \mathbb{R}\} \\ &= \sup\{|\varphi(g).0| : \varphi \in \mathrm{Harm}(G)\},\end{aligned}$$

which is finite by compactness.

6.3 Criteria for membership in $\mathcal{C}$

This section is devoted to the proof of Theorem 6.3.4, which is the main criterion used in subsequent sections to decide membership in $\mathcal{C}$. Subsection 6.3.1 introduces the spaces of irreducible actions of interest, Subsection 6.3.2 proves preliminary lemmas enabling the use of the Glimm-Effros dichotomy, and Subsection 6.3.3 proves Theorem 6.3.4. In this section, G is always a countable group.

6.3.1 Spaces of actions

The group $\mathrm{Homeo}_0(\mathbb{R})$ is naturally identified with $\mathrm{Homeo}_0([0,1])$, and a result of R. Arens [Are46, Theorems 1 & 5] implies that this identification is a homeomorphism when both groups are given the compact-open topology. Hence $\mathrm{Homeo}_0(\mathbb{R})$ with the compact-open topology is a Polish group, and the inequality

$$\sup_{z\in[x,y]} |f(z) - g(z)| \leq \max(f(y), g(y)) - \min(f(x), g(x)),$$

valid for $x \leq y$ and $f, g \in \mathrm{Homeo}_0(\mathbb{R})$, shows that this topology coincides with the topology of pointwise convergence.

Fix G a countable group and endow $\mathrm{Hom}(G, \mathrm{Homeo}_0(\mathbb{R}))$ with the induced topology from the product topology on $\mathrm{Homeo}_0(\mathbb{R})^G$. Define:

- $\mathrm{Rep}_{\mathrm{cc}}(G) \subseteq \mathrm{Rep}_{\mathrm{irr}}(G)$ the space of cocompact actions of G,
- $\mathrm{Rep}_{\min}(G), \mathrm{Rep}_{\mathrm{cyc}}(G) \subseteq \mathrm{Rep}_{\mathrm{cc}}(G)$ the space of minimal actions of G and the space of cyclic actions of G, and
- $\mathrm{Rep}_{\mathrm{III}}(G), \mathrm{Rep}_{\mathrm{cent}}(G) \subseteq \mathrm{Rep}_{\min}(G)$ the space of minimal type III actions of G and the space of minimal actions of G with non-trivial centralizer.

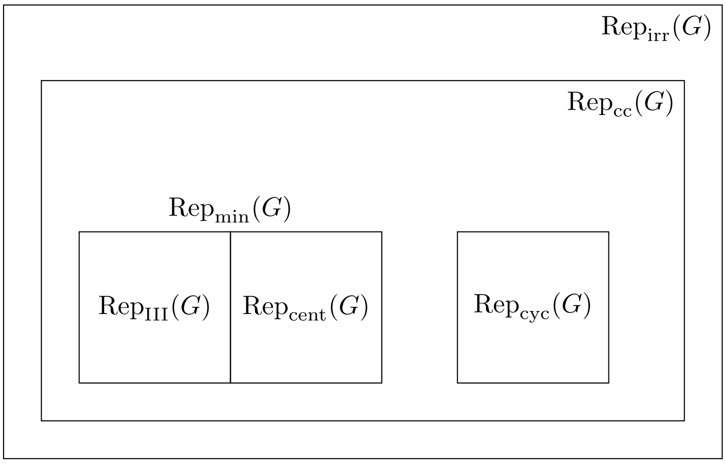

Figure 6.1

The inclusions between these spaces are summarized in Figure 6.1.

By writing

$$\mathrm{Rep}_{\mathrm{min}}(G) = \bigcap_{\substack{q\in\mathbb{Q},n\in\mathbb{N}_+,\\ m\in\mathbb{Z}}} \bigcup_{\substack{F\subseteq G\\ \text{finite}}} \{\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R}) : \varphi(F).(q-1/n, q+1/n) \supseteq [m, m+1]\}$$

and

$$\mathrm{Rep}_{\mathrm{III}}(G) = \bigcap_{\substack{p,q,r,s\in\mathbb{Q}\\ p<q,\, r<s}} \bigcup_{g\in G} \{\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R}) : \varphi(g).[p,q] \subseteq (r,s)\}$$

we see that $\mathrm{Rep}_{\mathrm{min}}(G)$ and $\mathrm{Rep}_{\mathrm{III}}(G)$ are G_δ sets inside $\mathrm{Hom}(G, \mathrm{Homeo}_0(\mathbb{R}))$, so they are also Polish for the induced topology [Kec95, (3.11)].

We consider the continuous action of $\mathrm{Homeo}_0(\mathbb{R})$ on $\mathrm{Rep}_{\mathrm{min}}(G)$ by conjugation, defined by $(f.\varphi)(g) = f \circ \varphi(g) \circ f^{-1}$ for every $f \in \mathrm{Homeo}_0(\mathbb{R})$, $\varphi \in \mathrm{Rep}_{\mathrm{min}}(G)$ and $g \in G$. The stabilizer $\mathrm{Stab}_{\mathrm{Homeo}_0(\mathbb{R})}(\varphi)$ of an action $\varphi \in \mathrm{Rep}_{\mathrm{min}}(G)$ coincides with its centralizer $\mathrm{Cent}_\varphi(G)$, which is thus closed in $\mathrm{Homeo}_0(\mathbb{R})$.

6.3.2 Approximate conjugacies

Lemma 6.3.1. *Consider minimal actions $\varphi, \psi \in \mathrm{Rep}_{\mathrm{min}}(G)$ such that there are sequences $(f_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$ and $(\varphi_n)_{n\geq 0} \subseteq \mathrm{Rep}_{\mathrm{min}}(G)$ with $\lim_{n\to\infty} \varphi_n = \varphi$ and $\lim_{n\to\infty} f_n.\varphi_n = \psi$. If there exists an $x \in \mathbb{R}$ such that $(f_n^{-1}(x))_{n\geq 0}$ admits a limit point $y \in \mathbb{R}$, then there exists $f \in \mathrm{Homeo}_0(\mathbb{R})$ with $f.\varphi = \psi$ and $f(y) = x$.*

Proof. We claim that for every $g \in G$, the inequality $\varphi(g).y > y$ implies $\psi(g).x \geq x$: indeed, if $\varphi(g).y > y$ then for sufficiently large $n \in \mathbb{N}$ we have $\varphi_n(g).y > y$ and $\varphi_n(g).\big(f_n^{-1}(x)\big) > f_n^{-1}(x)$, so $(f_n.\varphi_n)(g).x > x$ and hence $\psi(g).x \geq x$. It also follows that $\varphi(g).y < y$ implies that $\psi(g).x \leq x$, and in particular $\varphi(g).y = y$ implies that $\psi(g).x = x$. Hence the correspondence $\varphi(g).y \mapsto \psi(g).x$

is a well-defined, non-decreasing equivariant map from $\varphi(G).x \to \psi(G).y$. By [BMBRT25, Lemma 2.3], this map extends to a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ conjugating φ to ψ. ☐

Proposition 6.3.2. *The orbit equivalence relation*

$$E_G^{\min} = \{(\varphi, \psi) \in \mathrm{Rep}_{\min}(G) \times \mathrm{Rep}_{\min}(G) : \text{there exists } f \in \mathrm{Homeo}_0(\mathbb{R}) \text{ such that } f.\varphi = \psi\}$$

is F_σ in $\mathrm{Rep}_{\min}(G) \times \mathrm{Rep}_{\min}(G)$.

Proof. For $k \in \mathbb{N}_+$ define F_k as the set of $(\varphi, \psi) \in E_G^{\min}$ such that there exists $f \in \mathrm{Homeo}_0(\mathbb{R})$ with $\left|f^{-1}(0)\right| \leq k$ and $f.\varphi = \psi$. We will prove that each F_k is closed in $\mathrm{Rep}_{\min}(G) \times \mathrm{Rep}_{\min}(G)$, implying the desired conclusion since $E_G^{\min} = \bigcup_{k \geq 0} F_k$.

Let $((\varphi_n, f_n.\varphi_n))_{n \geq 0} \subseteq F_k$ be a sequence such that $\lim_{n\to\infty} \varphi_n = \varphi$ and $\lim_{n\to\infty} f_n.\varphi_n = \psi$ for some $(\varphi, \psi) \in E_G^{\min}$. Up to considering a subsequence, we may assume that there exists $y \in [-k, k]$ such that $\lim_{n\to\infty} f_n^{-1}(0) = y$. By Lemma 6.3.1 we produce $f \in \mathrm{Homeo}_0(\mathbb{R})$ such that $f(y) = 0$ and $f.\varphi = \psi$, so $(\varphi, \psi) \in F_k$. ☐

Remark. Proposition 6.3.2 implies that the conjugacy relation on $\mathrm{Rep}_{\min}(G)$ and on all its subspaces we have considered are Borel. This conclusion can be obtained directly through an argument close to [JJ25, Lemma 6.5]: by [BK96, Theorem 7.1.2] this is equivalent to seeing that the map

$$C\colon \varphi \in \mathrm{Rep}_{\min}(G) \mapsto \mathrm{Stab}_{\mathrm{Homeo}_0(\mathbb{R})}(\varphi) = \mathrm{Cent}_\varphi(G) \in \mathrm{Sub}(\mathrm{Homeo}_0(\mathbb{R}))$$

is Borel, where $\mathrm{Sub}(\mathrm{Homeo}_0(\mathbb{R}))$ is the standard Borel space of closed subgroups of $\mathrm{Homeo}_0(\mathbb{R})$, endowed with the Effros Borel structure generated by the sets

$$B_U = \{H \in \mathrm{Sub}(\mathrm{Homeo}_0(\mathbb{R})) : U \cap H \neq \varnothing\}$$

for open $U \subseteq \mathrm{Homeo}_0(\mathbb{R})$ (see [Mal11, Proposition 1]).

Write

$$C^{-1}(B_U) = \{\varphi \in \mathrm{Rep}_{\min}(G) : U \cap \mathrm{Cent}_\varphi(G) \neq \varnothing\} = \pi(B)$$

where $B = \{(\varphi, g) \in \mathrm{Rep}_{\min}(G) \times U : g \in \mathrm{Cent}_\varphi(G)\}$ and $\pi\colon \mathrm{Rep}_{\min}(G) \times U \to \mathrm{Rep}_{\min}(G)$ is the projection onto the first coordinate. The space $\mathrm{Rep}_{\min}(G) \times U$ is Polish and B is Borel, and moreover for every $\varphi \in \mathrm{Rep}_{\min}(G)$ the section $B \cap \pi^{-1}(\varphi)$ is homeomorphic to $\mathrm{Cent}_\varphi(G) \cap U$. Since $\mathrm{Cent}_\varphi(G)$ is homeomorphic either to a point, $\mathbb{Z}$ or $\mathbb{R}$, we have that $B \cap \pi^{-1}(\varphi)$ is always K_σ, so Theorem 6.2.4 implies that $\pi(B)$ is Borel. We conclude that C is a Borel map.

6.3.3 Almost centralizing sequences and groups in $\mathcal{C}$

Definition 6.3.3. *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$. We say that a sequence $(f_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$* almost centralizes *$\varphi$ if $\lim_{n\to\infty} f_n.\varphi = \varphi$.*

The following will be our main tool to decide membership to $\mathcal{C}$.

Theorem 6.3.4. *For a countable group G, the following are equivalent:*

i. $G \in \mathcal{C}$*;*

ii. for every $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$*, every sequence* $(f_n)_{n\geq 0}$ *almost centralizing* φ *tends to* $\mathrm{id}_{\mathbb{R}}$*;*

iii. for every $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$ *and every sequence* $(f_n)_{n\geq 0}$ *almost centralizing* φ*, there exists* $z \in \mathbb{R}$ *such that* $(f_n^{-1}(z))_{n\geq 0}$ *is bounded.*

If G is finitely generated, then these conditions are also equivalent to:

iv. for every $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G) \cap \mathrm{Harm}(G)$ *and* $(t_n)_{n\geq 0} \subseteq \mathbb{R}$ *such that* $\lim_{n\to\infty} \Psi^{t_n}(\varphi) = \varphi$ *we have* $\lim_{n\to\infty} t_n = 0$*.*

Proof. The equivalence between (i) and (iv) follows from Proposition 6.2.6. We now concentrate on the other equivalences.

Notice that the conjugacy relation on $\mathrm{Rep}_{\mathrm{min}}(G)$ is precisely the orbit equivalence relation E_G^{min} of $\mathrm{Homeo}_0(\mathbb{R})$ acting on $\mathrm{Rep}_{\mathrm{min}}(G)$ by conjugation. Since E_G^{min} is F_σ by Proposition 6.3.2, the Glimm-Effros dichotomy shows that E_G^{min} is smooth if and only if for every $\varphi \in \mathrm{Rep}_{\mathrm{min}}(G)$, the orbital map

$$\Psi_\varphi \colon f \in \mathrm{Homeo}_0(\mathbb{R})/\mathrm{Cent}_\varphi(G) \mapsto f.\varphi \in \mathrm{Orb}_{\mathrm{Homeo}_0(\mathbb{R})}(\varphi)$$

is a homeomorphism (it is always continuous). Thus if G is in $\mathcal{C}$, then sequence $(f_n)_{n\geq 0}$ almost centralizing some $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$ must verify $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$, and in particular all the sequences $(f_n^{-1}(z))_{n\geq 0}$, $z \in \mathbb{R}$ are bounded. This proves that (i)⇒(ii)⇒(iii).

To prove the remaining implication (iii)⇒(i), suppose that every action in $\mathrm{Rep}_{\mathrm{III}}(G)$ satisfies the condition in the statement of the proposition, and let $\varphi \in \mathrm{Rep}_{\mathrm{min}}(G)$. We will show that Ψ_φ^{-1} is continuous, and this suffices to prove that $G \in \mathcal{C}$ by the previous paragraph.

Suppose first that $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$, and let $(f_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$ with $\lim_{n\to\infty} f_n.\varphi = \varphi$. Then there is a $z \in \mathbb{R}$ such that $(f_n^{-1}(z))_{n\geq 0}$ is bounded by the hypothesis. Fix $x \in \mathbb{R}$ and let $g \in G$ such that $\varphi(g).x < z$. Then for large enough $n \in \mathbb{N}$ we have $f_n \circ \varphi(g) \circ f_n^{-1}(x) < z$, or $f_n^{-1}(x) < \varphi(g^{-1}) \circ f_n^{-1}(z)$. Hence $(f_n^{-1}(x))_{n\geq 0}$ is bounded above, and a symmetrical argument shows that it is also bounded below.

Let $y \in \mathbb{R}$ be a limit point of the sequence $(f_n^{-1}(x))_{n\geq 0}$. Lemma 6.3.1 shows that there is a $f \in \mathrm{Homeo}_0(\mathbb{R})$ such that $f(x) = y$ and $f.\varphi = \varphi$. Since φ is of type III, f must be $\mathrm{id}_{\mathbb{R}}$ and $y = x$. Thus $\lim_{n\to\infty} f_n^{-1}(x) = x$, and as this is true for every $x \in \mathbb{R}$, we have $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$. Thus Ψ_φ^{-1} is continuous in this case.

Suppose that $\varphi \in \mathrm{Rep}_{\mathrm{cent}}(G)$ instead, and let $(f_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$ with $\lim_{n\to\infty} f_n.\varphi = \varphi$. If φ is of type I, then we can find $(c_n)_{n\geq 0} \subseteq \mathrm{Cent}_\varphi(G)$ such that $\lim_{n\to\infty} c_n \circ f_n^{-1}(0) = 0$. Suppose instead that φ is of type II. Let c be the generator of $\mathrm{Cent}_\varphi(G)$ with $c(0) > 0$, and take $(c_n)_{n\geq 0} \subseteq \mathrm{Cent}_\varphi(G)$ such that $c_n \circ f_n^{-1}(0) \in [c^{-1}(c(0)/2), c(0)/2)$ for every $n \in \mathbb{N}$. In any case, we have

$$\lim_{n\to\infty} (f_n \circ c_n^{-1}).\varphi = \lim_{n\to\infty} f_n.\varphi = \varphi,$$

and Lemma 6.3.1 shows that any limit point y of $c_n \circ f_n^{-1}(0)$ is of the form $f(0)$ for some function $f \in \mathrm{Cent}_\varphi(G)$. Hence $y = 0$, and since this is true for any limit point we deduce that

$$\lim_{n\to\infty} c_n \circ f_n^{-1}(0) = 0$$

also in this case.

Now assume towards a contradiction that for some $x > 0$ we have $\limsup_{n\to\infty} c_n \circ f_n^{-1}(x) > x$, and upon passing to a subsequence we may assume that $\liminf_{n\to\infty} c_n \circ f_n^{-1}(x) > z$ for some $z > x$. Take $0 < \varepsilon < x$ small enough so that there is $g \in G$ with $\varphi(g).[-\varepsilon, \varepsilon] \subseteq (x, z)$. But for n large enough we have that $c_n \circ f_n^{-1}(0) \in [-\varepsilon, \varepsilon]$, so $\varphi(g) \circ c_n \circ f_n^{-1}(0) \in (x, z)$. For n large enough we conclude that

$$(f_n.\varphi)(g).0 = f_n \circ c_n^{-1} \circ \varphi(g) \circ c_n \circ f_n^{-1}(0) < x,$$

contradicting the assumption $\lim_{n\to\infty} f_n.\varphi(g) = \varphi(g)$. Hence $\limsup_{n\to\infty} c_n \circ f_n^{-1}(x) \leq x$ for every $x > 0$.

Similarly, assume that for some $x > 0$ we have $\liminf_{n\to\infty} c_n \circ f_n^{-1}(x) < x$, so upon passing to a subsequence we have $\limsup_{n\to\infty} c_n \circ f_n^{-1}(x) < z$ for some $z \in (0, x)$. Take $0 < \varepsilon < z$ small enough so that there is $g \in G$ with $\varphi(g).[-\varepsilon, \varepsilon] \subseteq (z, x)$. Again, for n large enough we have $c_n \circ f_n^{-1}(0) \in [-\varepsilon, \varepsilon]$ and $\varphi(g) \circ c_n \circ f_n^{-1}(0) \subseteq (x, z)$. We conclude that for n large the inequality

$$(f_n.\varphi)(g).0 = f_n \circ c_n^{-1} \circ \varphi(g) \circ c_n \circ f_n^{-1}(0) > x$$

holds, contradicting $\lim_{n\to\infty} f_n.\varphi(g) = \varphi(g)$. Hence $\lim_{n\to\infty} c_n \circ f_n^{-1}(x) = x$.

A symmetrical argument gives $\lim_{n\to\infty} c_n \circ f_n^{-1}(x) = x$ for all $x < 0$ also. We deduce that $\lim_{n\to\infty} c_n \circ f_n^{-1} = \mathrm{id}_{\mathbb{R}}$ in $\mathrm{Homeo}_0(\mathbb{R})$ and that $\lim_{n\to\infty} f_n \mathrm{Cent}_\varphi(G) = \mathrm{Cent}_\varphi(G)$. We conclude that Ψ_φ^{-1} is continuous. □

The previous theorem and Theorem 6.2.1 imply the following, which was known for finitely generated groups as a consequence of Proposition 6.2.6.

Corollary 6.3.5. *If G does not admit any minimal proximal action on the line (in particular, if G has no non-abelian free semigroups), then $G \in \mathcal{C}$.*

6.4 Actions on the line and commensurated subgroups

In this section we collect several structural statements for minimal actions on the line of a countable group admitting a commensurated subgroup, to be used in the next section. These are natural generalisations of the corresponding statements for normal subgroups. Subsection 6.4.2 describes the case when the action of the commensurated subgroup is of type I, and Subsection 6.4.3 describes the case when it admits no minimal set. In this section G is always a countable group.

6.4.1 Commensurated subgroups and isolator

Recall that two subgroups H, K of a group G are *commensurate* if $H \cap K$ has finite index in G. The subgroup H is *isolated* if whenever $g^n \in H$ for some $g \in G$, $n \in \mathbb{Z}$, then $g \in H$. The smallest isolated subgroup containing H is called the *isolator* of H, and is denoted by $\mathrm{I}(H)$. The following straightforward lemma explains why this notion arises here.

Lemma 6.4.1. *Let H be a commensurated subgroup of G. Then its isolator $\mathrm{I}(H)$ does not depend on the choice of H in its commensurability class, and it is equal to the smallest normal subgroup of G containing H and such that $G/\mathrm{I}(H)$ is torsion free.*

In particular, every $\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R})$ such that $\varphi\big|_H$ is trivial factors through $G/\mathrm{I}(H)$.

6.4.2 Conrad homomorphisms and affine actions

We first describe actions semiconjugate to actions by translations, following [Nav11, Section 2.2]. An action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ is of type I if and only if it preserves a non-trivial Radon measure ν on $\mathbb{R}$. In this, case the map $\tau_\nu\colon G \to \mathbb{R}$ given by

$$g \in G \mapsto \tau_\nu(g) = \begin{cases} \nu[x, \varphi(g).x) & \text{if } x \le \varphi(g).x \\ -\nu[\varphi(g).x, x) & \text{if } x > \varphi(g).x \end{cases} \tag{6.4.1}$$

is well defined and does not depend on $x \in \mathbb{R}$. It is actually a group morphism called the *Conrad homomorphism associated to* φ, which is unique up to positive rescaling.

Proposition 6.4.2 (see [Nav11, Section 2.2])**.** *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ be an action of type I with a minimal set Λ, and let ν be a non-trivial Radon measure on $\mathbb{R}$ invariant under $\varphi(G)$.*

i. The action φ is semiconjugate to the action by translations defined by τ_ν, and $\ker(\tau_\nu)$ coincides with the elements $h \in H$ such that $\varphi(h)$ is trivial on Λ. In particular, τ is trivial if and only if φ has a global fixed point.

ii. If ν' is another non-zero Radon measure preserved by $\varphi(G)$ there exists a $\kappa > 0$ such that $\tau_{\nu'} = \kappa\tau_\nu$. If moreover $\tau_\nu(G)$ is dense in $\mathbb{R}$, then actually $\nu' = \kappa\nu$.

A similar statement is true for actions semiconjugate to affine actions: let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ and let ν be a Radon measure on $\mathbb{R}$. We say that φ *preserves the projective class of* ν if for every $g \in G$ there exists $\kappa(g) > 0$ such that the measures $\nu(\varphi(g).\cdot)$ and $\kappa(g)\nu(\cdot)$ are equal. In this case, the map

$$g \in G \mapsto \tau_\nu(g) = \begin{cases} \nu[0, \varphi(g).0) & \text{if } 0 \le \varphi(g).0 \\ -\nu[\varphi(g).0, 0) & \text{if } \varphi(g).0 < 0 \end{cases} \tag{6.4.2}$$

satisfies $\tau_\nu(g_1g_2) = \tau_\nu(g_1) + \kappa(g_1)\tau_\nu(g_2)$ for all $g_1, g_2 \in G$. Any action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ semiconjugate to an action factoring through the affine group

$$\mathrm{Aff}(\mathbb{R}) = \{x \mapsto ax + b : a \in \mathbb{R}_+,\, b \in \mathbb{R}\}$$

preserves the projective class of the measure given by the pullback through the semiconjugacy of the Lebesgue measure on $\mathbb{R}$. Conversely, if φ preserves the projective class of a Radon measure ν on $\mathbb{R}$, then φ is semiconjugate to the affine action given by

$$g \in G \mapsto \big(x \in \mathbb{R} \mapsto \kappa(g)(x - \tau_\nu(g^{-1}))\big)\,. \tag{6.4.3}$$

where κ is such that $\varphi(g)^*\nu = \kappa(g)\nu$ for all $g \in G$ and τ_ν is defined as in (6.4.2), see [Nav11, Proposition 1.2.2].

Hence any minimal action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ such that $\varphi\big|_N$ is minimal and of type I for some normal subgroup $N \subseteq G$ is actually affine, since the normality of N implies that $\varphi(G)$ preserves the projective class of any $\varphi(N)$-invariant measure on $\mathbb{R}$. We now verify a version of this statement in the case when N is only commensurated in G. The starting point is the following observation.

Proposition 6.4.3. *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ be a minimal action and let $H \subseteq G$ be a commensurated subgroup. If $\varphi\big|_H$ has a global fixed point, then $\varphi\big|_H$ is trivial.*

Proof. Let $g \in G$ and $h \in gHg^{-1}$, so there exists an $n \in \mathbb{N}_+$ such that $h^n \in H$. If $x \in \mathrm{Fix}_\varphi(H)$ then $\varphi(h^n)(x) = x$, and thus $\varphi(h)(x) = x$. This shows that $\mathrm{Fix}_\varphi(H) \subseteq \mathrm{Fix}_\varphi(gHg^{-1})$ and a symmetric argument gives $\mathrm{Fix}_\varphi(H) = \mathrm{Fix}_\varphi(gHg^{-1})$, so $\varphi(g).\mathrm{Fix}_\varphi(H) = \mathrm{Fix}_\varphi(H)$. Thus $\mathrm{Fix}_\varphi(H)$ is closed, $\varphi(G)$-invariant and nonempty, and by minimality it must be all $\mathbb{R}$. □

Given a commensurated subgroup $H \subseteq G$ and $g \in G$, we write $\triangle(g)\colon H \cap g^{-1}Hg \to H$ for conjugation by g.

Lemma 6.4.4. *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ be a minimal action such that $\varphi\big|_H$ is irreducible and of type I for some commensurated subgroup $H \subseteq G$. Let $\tau\colon H \to \mathbb{R}$ be the Conrad morphism for H associated to $\varphi\big|_H$.*

i. There exists a group morphism $\kappa\colon G \to \mathbb{R}^*_+$ such that $\tau \circ \triangle(g)(h) = \kappa(g)\tau(h)$ for all $g \in G$ and $h \in H \cap g^{-1}Hg$.

ii. $\ker(\tau)$ *is commensurated in G and acts trivially through φ.*

iii. If $\varphi(H)$ is non-cyclic, then φ is conjugate to an affine action φ_{aff} as in (6.4.3). *Moreover, if φ is semiconjugate to an affine action φ_{aff}, the element $\varphi_{\mathrm{aff}}(h)$ is a translation whenever $h \in G$ belongs to a subgroup $L \subseteq G$ commensurate with H.*

Proof. We first aim to find, for a fixed $g \in G$, a $\kappa(g) > 0$ verifying $\tau \circ \triangle(g)(h) = \kappa(g)\tau(h)$ for all $h \in H \cap g^{-1}Hg$.

Since $\varphi\big|_H$ is of type I, there exists an H-invariant Radon measure ν on $\mathbb{R}$ as in (6.4.2) such that $\tau = \tau_\nu$. Define ν_g as the Radon measure on $\mathbb{R}$ given by $\nu_g(\cdot) = \nu(\varphi(g)\,\cdot)$. It is straightforward to see that the measure ν_g is $H \cap g^{-1}Hg$-invariant, so there exists a $\kappa(g) > 0$ such that $\tau_{\nu_g} = \kappa(g)\tau_\nu$ on $H \cap g^{-1}Hg$ by Proposition 6.4.2. We conclude that when $h \in H \cap g^{-1}Hg$ is such that $\tau(h) \geq 0$, then

$$\tau_{\nu_g}(h) = \nu_g[\varphi(g^{-1}).x, \varphi(h) \circ \varphi(g^{-1}).x) = \nu[x, \varphi(ghg^{-1}).x) = \tau \circ \triangle(g)(h),$$

and the same equality holds by linearity whenever $h \in H \cap g^{-1}Hg$ is such that $\tau(h) < 0$.

We now show that $\ker(\tau)$ is commensurated in G. Take $g \in G$ and set $K = \ker(\tau)$. Consider the map

$$\iota\colon h(K \cap g^{-1}Kg) \in K/(K \cap g^{-1}Kg) \mapsto h(H \cap g^{-1}Hg) \in H/(H \cap g^{-1}Hg)$$

which is well defined since $K \subseteq H$. Notice that $H \cap g^{-1}Hg \cap K \subseteq K \cap g^{-1}Kg$: indeed, if $h \in H \cap g^{-1}Hg \cap K$, then

$$\tau(ghg^{-1}) = \tau \circ \triangle(g)(h) = \kappa(g)\tau(h) = 0,$$

hence $h \in g^{-1}Kg$ and $h \in K \cap g^{-1}Kg$. Since $H \cap g^{-1}Hg \cap K \subseteq K \cap g^{-1}Kg$, the map ι is injective. Thus $K \cap g^{-1}Kg$ has finite index in K and K is commensurated in G.

By Proposition 6.4.2 the set $\mathrm{Fix}_\varphi(K)$ is non-empty. But the subgroup K is commensurated in G, hence $\mathrm{Fix}_\varphi(K) = \mathbb{R}$, showing (ii). By hypothesis $\varphi\big|_H$ is non-trivial, so $K = \ker(\tau)$ does not coincide with H and thus τ is non-trivial. We conclude that $\kappa\colon G \to \mathbb{R}_+^*$ is a morphism, showing (i).

We now prove (iii). Again Proposition 6.4.2 shows that $\nu_g = \kappa(g)\nu$ (that is, φ preserves the projective class of ν) when $\tau_\nu(H)$ is dense. Since $\varphi(H)$ is isomorphic to $\tau(H)$ by (ii), this happens if and only if $\varphi(H)$ is non-cyclic. If this is so, then φ must be semiconjugate (actually conjugate by minimality of φ) to an affine action.

For the second statement of (iii), notice that if $h \in L$ where $L \subseteq G$ is a subgroup commensurate with H, then there is an $n \in \mathbb{N}_+$ with $h^n \in H$. The action φ_{aff} preserves the projective class of the Lebesgue measure on $\mathbb{R}$, so we have a morphism $\kappa_{\mathrm{Leb}}\colon G \to \mathbb{R}_+^*$ such that $\varphi_{\mathrm{aff}}(g)$ rescales Lebesgue measure by $\kappa_{\mathrm{Leb}}(g)$. Thus

$$\kappa_{\mathrm{Leb}}(h)^n \mathrm{Leb} = \varphi_{\mathrm{aff}}(h^n)^* \mathrm{Leb} = \mathrm{Leb}$$

and $\kappa_{\mathrm{Leb}}(h) = 1$. We conclude that $\varphi_{\mathrm{aff}}(h)$ preserves Leb, that is, $\varphi_{\mathrm{aff}}(h)$ is a translation. □

6.4.3 Laminar actions

Two open and bounded intervals $I, J \subseteq \mathbb{R}$ are said to be *crossed* if $I \cap J \neq \varnothing$ and neither $I \subseteq J$ or $J \subseteq I$. A *prelamination* is a collection $\mathcal{L}$ of open, bounded and non-empty intervals of $\mathbb{R}$ that pairwise do not cross. A *lamination* is a prelamination $\mathcal{L}$ that is closed when seen as a subset of $\{(x, y) \in \mathbb{R}^2 : x < y\}$ with its natural topology, and its elements are called *leaves*. An action $\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R})$ is said to be *laminar* if it preserves a lamination $\mathcal{L}$ that is *covering*, that is, that contains an increasing exhaustion of $\mathbb{R}$.

Laminar actions on the line are studied in [BMBRT24]. Here we will only recall that covering laminations immediately arise in group actions on the line that admit no minimal sets. A *wandering interval* for an action $\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R})$ is an open bounded and non-empty interval $I \subseteq \mathbb{R}$ such that for all $g \in G$, either $\varphi(g).I = I$ or $\varphi(g).I \cap I = \varnothing$. An *irreducible wandering interval* for $\varphi(G)$ is a wandering interval I for $\varphi(H)$ such that $\mathrm{Stab}_{\varphi(H)}(I)$ acts on I without global fixed points. We denote by $\mathcal{W}_\varphi(G)$ the set of irreducible wandering intervals of $\varphi(G)$.

Lemma 6.4.5 ([BMBRT24, Lemma 8.3.2]). *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ be an action that does not admit a minimal set. Then $\mathcal{W}_\varphi(G)$ is a $\varphi(G)$-invariant covering prelamination.*

A laminar action $\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R})$ can be thought as coming from the action of G on a (not necessarily simplicial) tree $\mathcal{T}$ that fixes a point in the boundary of $\mathcal{T}$. An analogue of the general classification of group actions on trees by J. Tits [Tit70] is still true for laminar actions, but only two cases arise when the action is assumed irreducible. Call a subset $\mathcal{X}$ of a lamination $\mathcal{L}$ *cofinal* if for every $l \in \mathcal{L}$ there exists $I \in \mathcal{X}$ with $l \subseteq I$.

Proposition 6.4.6 ([BMBRT24, Proposition 8.1.10])**.** *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ be a laminar and irreducible action. Let $\mathcal{L}$ be a covering lamination preserved by φ. Then exactly one of the following hold.*

- *There is a cofinal set of wandering intervals for $\varphi(G)$ in $\mathcal{L}$. In this case there is no minimal set for $\varphi(G)$, and for every finitely generated $H \subseteq G$ the set $\{l \in \mathcal{L} : \varphi(H).l = l\}$ is cofinal in $\mathcal{L}$ and $\mathrm{Fix}_{\varphi}(H) \subseteq \mathbb{R}$ is unbounded in both directions. The action φ is said to be* horocyclic*.*

- *There exists an $l \in \mathcal{L}$ with cofinal $\varphi(G)$-orbit. In this case there exists a unique (non-discrete) minimal set for $\varphi(G)$ and φ is proximal. The action φ is said to be* focal*.*

If $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ is an action such that $\varphi\big|_N$ admits no minimal set for some normal subgroup $N \subseteq G$ then $\mathcal{W}_{\varphi}(N)$ is also $\varphi(G)$-invariant, and hence φ is laminar. The same statement holds when N is only commensurated.

Lemma 6.4.7. *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ such that $\varphi\big|_H$ does not admit a minimal set for some commensurated subgroup $H \subseteq G$. Then $\mathcal{W}_{\varphi}(H)$ is $\varphi(G)$-invariant, and hence φ is laminar.*

Proof. Since $\varphi\big|_H$ does not admit a minimal set, $\varphi\big|_H$ must be irreducible and Lemma 6.4.5 shows that $\mathcal{W}_{\varphi}(H)$ is a covering prelamination.

We claim that if $I \in \mathcal{W}_{\varphi}(H)$ and $J \in \varphi(g).\mathcal{W}_{\varphi}(H) = \mathcal{W}_{\varphi}(gHg^{-1})$ for some $g \in G$, then I and J do not cross. Indeed, suppose towards a contradiction that $I = (a, b)$ and $J = (c, d)$ cross, so we may assume that $a < c < b < d$. The action of $\mathrm{Stab}_{\varphi(H)}(I)$ is fixed-point-free, so there exists $h \in H$ such that $\varphi(h).I = I$ and $\varphi(h).c \neq c$. If $n \in \mathbb{N}_+$ is such that $h^n \in gHg$, then $\varphi(h^n).J$ cannot be disjoint from J because $\varphi(h^n).b = b$. Thus $\varphi(h^n).J = J$, which contradicts $\varphi(h^n).c \neq c$. We conclude that I and J do not cross.

Now let $I \in \mathcal{W}_{\varphi}(H)$, $g \in G$. We claim that $\varphi(g).I \in \mathcal{W}_{\varphi}(H)$: take $h \in H$ and $n \in \mathbb{N}_+$ such that $g^{-1}h^n g \in H$. Then either $\varphi(g^{-1}h^n g).I = I$ or $\varphi(g^{-1}h^n g).I \cap I = \varnothing$. If the first option is satisfied we also have $\varphi(g^{-1}hg).I = I$. If the second option is satisfied, suppose that $\varphi(g^{-1}hg).I \cap I \neq \varnothing$. Then $\varphi(g^{-1}hg).I$ and I must be crossed, contradicting the previous claim, so $\varphi(g^{-1}hg).I \cap I = \varnothing$. In any case, we conclude that $\varphi(h)\,(\varphi(g).I) = \varphi(g).I$ or $\varphi(h)\,(\varphi(g).I) \cap \varphi(g).I = \varnothing$ and that $\varphi(g).I$ is also wandering. Since $\mathrm{Stab}_{\varphi(gHg^{-1})}(\varphi(g).I)$ has no fixed points in $\varphi(g).I$ and $\varphi(gHg^{-1}) \cap \varphi(H)$ has finite index in $\varphi(H)$, it follows that $\mathrm{Stab}_{\varphi(H)}(\varphi(g).I)$ has no fixed points in $\varphi(g).I$ either. Thus $\varphi(g).I \in \mathcal{W}_{\varphi}(H)$.

The closure of $\mathcal{W}_{\varphi}(H)$ in $\{(x, y) \in \mathbb{R}^2 : x < y\}$ gives a $\varphi(G)$-invariant covering lamination, so φ is laminar. □

6.5 Stability properties of $\mathcal{C}$

In this section we prove all the claimed stability properties of $\mathcal{C}$ and show some examples. Subsection 6.5.1 proves Theorem L, and Subsection 6.5.2 proves Corollaries 6.1.3, along with a more general theorem for fundamental groups of graphs of groups. We also introduce the necessary definitions from Bass-Serre theory.

6.5.1 Proof of Theorem L

In this subsection G will always denote a finitely generated group.

Lemma 6.5.1. *Let G be a finitely generated group. Let $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$ and $(f_n)_{n\geq 0}$ almost centralizing φ. Suppose that $H \subseteq G$ is a commensurated subgroup such that $\varphi\big|_H$ is non-trivial and is not of type III. Then $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$.*

Proof. First of all, $\varphi(H)$ cannot have an exceptional minimal set: suppose on the contrary that $\Lambda \subsetneq \mathbb{R}$ is such a minimal set and let $K \subseteq H$ be any normal finite-index subgroup. We claim that $\varphi(K)$ has a minimal set, which is Λ. Indeed, by reducing K we may assume that K is normal in H. The action of $\varphi(K)$ is cocompact since $\varphi(H)$ is cocompact and K has finite index in H, so $\varphi(K)$ has a minimal set Λ_K. It cannot be a discrete orbit, since in that case $\varphi(H)$ would preserve a union of discrete orbits. Thus Λ_K is unique, and by normality of K in H it is preserved by $\varphi(H)$. We conclude that $\Lambda_K = \Lambda$.

Now let $g \in G$, so by the previous paragraph the groups $\varphi(H), \varphi(H \cap gHg^{-1})$ and $\varphi(gHg^{-1})$ all contain a common normal finite index subgroup, hence share the same minimal set Λ. But $\varphi(g).\Lambda$ is also $\varphi(gHg^{-1})$-invariant, and hence contains a $\varphi(gHg^{-1})$-minimal set, which must be Λ by uniqueness. We conclude that $\varphi(g).\Lambda \supseteq \Lambda$, and since $g \in G$ was arbitrary we deduce that Λ is $\varphi(G)$-invariant. This contradicts the minimality of φ.

By Proposition 6.4.3 we have $\varphi\big|_H \in \mathrm{Rep}_{\mathrm{irr}}(H)$, and we will proceed to verify the conclusion according to whether $\varphi\big|_H$ is of type I, II or has no minimal set.

- Assume that $\varphi\big|_H$ is of type II. In this case $\varphi\big|_H$ cannot have a discrete orbit, and by the previous paragraph it cannot have an exceptional minimal set either, hence it must be minimal. We say that a sequence $(h_n)_{n\geq 0} \subseteq \mathrm{Homeo}_0(\mathbb{R})$ *contracts* a compact interval $I \subseteq \mathbb{R}$ if $h_n(I) \subseteq I$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{diam}(h_n(I)) = 0$. Thus the generator c of the centralizer $\mathrm{Cent}_\varphi(H)$ such that $c(0) > 0$ can be written as

  $$c(x) = \sup\{y \geq x : [x, y] \text{ is contracted by some sequence in } \varphi(H)\},$$

  see [Ghy01, Section 5.2].

  For any $g \in G$ we have, by definition, that

  $$\varphi(g) \circ c \circ \varphi(g^{-1})(x) = \sup\{y \geq x : \varphi(g^{-1}).[x, y] \text{ is contracted by some sequence in } \varphi(H)\}.$$

  Consider a sequence $(\varphi(h_n))_{n\geq 0} \subseteq \varphi(H)$ contracting an interval $[x, y]$. If $k \in \mathbb{N}_+$ is the index of $H \cap gHg^{-1}$ in H, then $(\varphi(h_n^k))_{n\geq 0}$ still contracts $[x, y]$, and every $\varphi(h_n^k)$ can be written as $\varphi(g\widetilde{h_n}g^{-1})$ for some $\widetilde{h_n} \in H$. Hence $(\widetilde{h_n})_{n\geq 0}$ contracts the interval $\varphi(g^{-1}).[x,y]$, so $c(x) \leq \varphi(g) \circ c \circ \varphi(g^{-1})(x)$ for any $x \in \mathbb{R}$. A symmetric argument replacing g by g^{-1} shows that $c = \varphi(g) \circ c \circ \varphi(g^{-1})$ and, since $g \in G$ was arbitrary, that φ has a non-trivial centralizer. This contradicts the fact that φ is of type III, so $\varphi\big|_H$ cannot be of type II.

We will assume that there exists an $x \in \mathbb{R}$ such that $(f_n^{-1}(x))_{n\geq 0}$ is unbounded and deduce a contradiction with this hypothesis in the remaining cases. This suffices: indeed since φ has trivial centralizer, Lemma 6.3.1 implies that the only finite accumulation point of $(f_n^{-1}(x))_{n\geq 0}$ can be x, hence if $(f_n^{-1}(x))_{n\geq 0}$ is bounded for every x then necessarily $f_n \to \mathrm{id}_{\mathbb{R}}$. We may assume that $\lim_{n\to\infty} f_n^{-1}(x) = \infty$. Fix a finite symmetric generating set $S \subseteq G$.

- Assume that $\varphi\big|_H$ is of type I. Let $\tau\colon H \to \mathbb{R}$ be a Conrad homomorphism associated to $\varphi\big|_H$ and let $\kappa\colon G \to \mathbb{R}_+^*$ as in Lemma 6.4.4. We claim that $\kappa(G) \neq \{1\}$. Suppose not, and notice that since $H \cap \bigcap_{s\in S} s^{-1}Hs$ has finite index in H, it contains an element h such that $\tau(h) > 0$ because $\varphi\big|_H$ is non-trivial. For every $s \in S$ we have

$$\tau(hsh^{-1}s^{-1}) = \tau(h) + \tau \circ \triangle(s)(h^{-1}) = \tau(h) - \kappa(s)\tau(h) = 0,$$

so by Lemma 6.4.4, (ii) the element $\varphi(hsh^{-1}s^{-1})$ is trivial. Hence $\varphi(h)$ is non-trivial and commutes with $\varphi(G)$, a contradiction.

Thus we can find $g \in G$ such that $\kappa(g) > 1$. Let $h \in H \cap gHg^{-1}$ such that $\varphi(h).x > \varphi(g).x$, so

$$U = \{\psi \in \mathrm{Rep}_{\mathrm{III}}(G) : \psi(h).x > \psi(g).x\}$$

is a neighborhood of φ in $\mathrm{Rep}_{\mathrm{III}}(G)$. We will show that $\varphi(h).y \leq \varphi(g).y$ for all sufficiently large $y \in \mathbb{R}$. Choose $r \in \mathbb{N}$ such that $\varphi(h^r g).s \geq \varphi(h).s$ for all $s \in [0, \varphi(h).0]$. Since $\kappa(g) > 1$ we can choose $N \in \mathbb{N}$ such that $\kappa(g)n - 1 \geq n + r$ for all $n \geq N$.

We claim that $\varphi(h).y < \varphi(g).y$ for all $y \geq \varphi(h^N).0$. Indeed, write $y = \varphi(h^n).t$ where $n \geq N$ and $t \in [0, \varphi(h).0]$. Denote by ν a Radon measure on $\mathbb{R}$ such that $\tau = \tau_\nu$ as in (6.4.1). Take $k = k(n,t) \in \mathbb{Z}$ such that

$$\varphi(g^{-1}h^{k-1}g).t \leq \varphi(h^n).t < \varphi(g^{-1}h^k g).t. \tag{6.5.1}$$

Then

$$n\tau(h) = \nu[t, \varphi(h^n).t) \leq \nu[t, \varphi(g^{-1}h^k g).t) = \tau \circ \triangle(g^{-1})(h^k) = k\kappa(g^{-1})\tau(h).$$

We conclude that $k \geq \kappa(g)n - 1 \geq n + r$ because $n \geq N$, and (6.5.1) implies

$$\varphi(g).y = \varphi(gh^n).t \geq \varphi(h^{k-1}g).t \geq \varphi(h^{n+r}g).t > \varphi(h^{n+1}).t = \varphi(h).y$$

as desired.

Since $\lim_{n\to\infty} f_n^{-1}(x) = \infty$ we deduce that $f_n.\varphi \notin U$ for sufficiently large $n \in \mathbb{N}$, which contradicts the assumption that $\lim_{n\to\infty} f_n.\varphi = \varphi$.

- Assume that $\varphi\big|_H$ does not admit a minimal set. By Lemma 6.4.7 the action φ preserves a covering lamination $\mathcal{L}$ composed of wandering intervals for $\varphi\big|_H$. Since G is finitely generated, the action φ is focal. Up to conjugating φ and the f_n we may assume that for every $g \in G$ the quantity $\sup\{|\varphi(g).y - y| : y \in \mathbb{R}\}$ is finite by Remark 6.2.7. Fix a maximal totally ordered (for inclusion) set $\mathcal{S} \subseteq \mathcal{L}$, which must be closed in $\mathcal{L}$ by maximality. The proofs of the following two claims follow some ideas in [BMBRT24, Section 15.1].

Claim 1. *For every finite subset F of G, there exists a leaf $l_F \in \mathcal{S}$ such that for all $l \in \mathcal{S}$ with $l \supseteq l_F$, we have $\varphi(F).l \subseteq \mathcal{S}$.*

Proof of the claim. Notice that since $\mathcal{L}$ is a lamination and $\mathcal{S}$ is maximal, if $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{S}$ are such that $l_1 \supseteq l_2$ then $l_1 \in \mathcal{S}$ too. Thus it suffices to find a leaf $l \in \mathcal{S}$ such that all its images under $\varphi(F)$ are over some element of $\mathcal{S}$.

Define the finite constant

$$\delta_F = \sup\{|\varphi(g).y - y| : g \in F,\, y \in \mathbb{R}\}.$$

Since $\mathcal{S}$ is maximal and $\mathcal{L}$ is covering, we can find leaves $l_1, l_2 \in \mathcal{S}$ such that $l_1 \supseteq l_2$ and the endpoints of l_1 are at distance at least $2\delta_F$ from the endpoints of l_2. Thus $\varphi(g).l_1$ still contains l_2 for any $g \in F$, so $\varphi(F).l_1 \subseteq \mathcal{S}$ and we are done. □

Claim 2. *There exist positive constants $C, D > 0$ such that for all $y > D$, the intervals $[y - C, y]$ and $[y, y + C]$ contain the right endpoint of a leaf of $\mathcal{S}$.*

Proof of the claim. Recall that $S \subseteq G$ is a fixed finite symmetric generating subset, and let $D > 0$ be greater than the right endpoint y_S of $l_S \in \mathcal{S}$. Define, as in the previous claim, the finite constant

$$\delta_S = \sup\{|\varphi(s).y - y| : s \in S,\, y \in \mathbb{R}\},$$

which must be positive since φ is irreducible.

For every $n \in \mathbb{N}$, pick $s_n \in S$ such that $\varphi(s_n s_{n-1} \cdots s_1).y_S - \varphi(s_{n-1} \cdots s_1).y_S$ is maximal. We have

$$\varphi(s_n s_{n-1} \cdots s_1).y_S - \varphi(s_{n-1} \cdots s_1).y_S \leq \delta_S$$

and $\lim_{n\to\infty} \varphi(s_n \cdots s_1).y_S = \infty$ by irreducibility of φ.

Now $\varphi(s_1).l_S \in \mathcal{S}$ by the definition of l_S, and since $\varphi(s_1).y_S > y_S$ and $\mathcal{S}$ is totally ordered we see that $\varphi(s_1).l_S \supsetneq l_S$. Similarly, for every $n \in \mathbb{N}$ we have that $\varphi(s_n \cdots s_1).l_S$ is a leaf of $\mathcal{S}$ that contains l_S. Thus any $y > D$ is such that $[y - \delta_S, y + \delta_S]$ contains a point $\varphi(s_m \cdots s_1).y_S$ for some $m \in \mathbb{N}$, which is the right endpoint of the leaf $\varphi(s_m \cdots s_1).l_S \in \mathcal{S}$. Setting $C = 2\delta_S$ and enlarging D if necessary we obtain the desired conclusion. □

The irreducibility of $\varphi\big|_H$ shows that there is an $h \in H$ with $\varphi(h).x > \varphi(s).x$ for every $s \in S$, and hence φ belongs to the open set

$$U = \{\psi \in \mathrm{Rep}_{\mathrm{III}}(G) : \psi(h).x > \psi(s).x \text{ for every } s \in S\}.$$

We will show that for all sufficiently large $y \in \mathbb{R}$ the inequality $\varphi(h).y \leq \varphi(s).y$ holds for at least one $s \in S$. Indeed, let y_h be the right endpoint of the leaf $l_{\{h\}\cup S}$ from Claim 1 applied to the finite set $\{h\} \cup S$. Notice that if $l \supseteq l_{\{h\}\cup S}$ then $\varphi(h).l \in \mathcal{S}$, but since $\mathcal{S}$ is totally ordered and l is $\varphi(H)$-wandering we conclude that $\varphi(h).l = l$ and $\varphi(h)$ fixes the extremities of l.

Let $C, D > 0$ be the constants from Claim 2, and take $y > \max(D, y_h)$. Thus the intervals $[y - C, y]$ and $[y, y + C]$ contain the right endpoints of leafs $l \supseteq l_{\{h\}\cup S}$ which must be fixed by h.

Using that $\mathcal{S}$ is closed we see that there are leaves $l_+ \supsetneq l_- \supseteq l_{\{h\}\cup S}$ such that l_- is maximal with the property that its right endpoint y_- is in $[y - C, y]$ and l_+ is minimal with the property that its right endpoint y_+ is in $[y, y + C]$. Since φ is focal and $\varphi(S).l_- \subseteq \mathcal{S}$ (because $l_- \supseteq l_{\{h\}\cup S}$) there is an $s \in S$ such that $\varphi(s).l_- \supseteq l_+$, so

$$\varphi(s).y \geq \varphi(s).y_- \geq y_+ = \varphi(h).y_+ \geq \varphi(h).y.$$

We conclude that if y is large enough we have $\varphi(h).y \leq \varphi(s).y$ for at least one $s \in S$.

We deduce that $f_n.\varphi \notin U$ for sufficiently large $n \in \mathbb{N}$, which contradicts the assumption that $\lim_{n\to\infty} f_n.\varphi = \varphi$. □

Recall that we denote by $\mathrm{I}(H)$ the isolator of a subgroup, see Subsection 6.4.1. The following proves Theorem L.

Theorem 6.5.2. *Let G be a finitely generated group. If G contains a commensurated subgroup H in $\mathcal{C}$ such that $G/\mathrm{I}(H)$ is in $\mathcal{C}$, then G is in $\mathcal{C}$.*

Proof. Let $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$ and $(f_n)_{n\geq 0}$ be an almost centralizing sequence.

If $\varphi\big|_H$ is trivial, then it is trivial on $\mathrm{I}(H)$ (see Lemma 6.4.1) and $\lim_{n\to\infty} f_n.\varphi = \varphi$ inside $\mathrm{Rep}_{\mathrm{III}}(G/\mathrm{I}(H))$. Thus $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$ because $G/\mathrm{I}(H) \in \mathcal{C}$. If $\varphi\big|_H \in \mathrm{Rep}_{\mathrm{III}}(H)$, then $(f_n)_{n\geq 0}$ almost centralizes $\varphi\big|_H$ and we conclude that $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$ because $H \in \mathcal{C}$. We conclude that $\lim_{n\to\infty} f_n = \mathrm{id}_{\mathbb{R}}$ in the remaining cases by Lemma 6.5.1.

We deduce that $G \in \mathcal{C}$ by Theorem 6.3.4. □

Corollary 6.5.3. *Let G be a finitely generated group. If G has a normal subgroup $N \in G$ such that $G/N \in \mathcal{C}$, then $G \in \mathcal{C}$.*

Given a family of groups $(H_i)_{i\in I}$, the direct sum $\bigoplus_I H_i$ is the subgroup of $\prod_I H_i$ of all $(h_i)_{i\in I}$ such that $h_i = e_{H_i}$ for all but finitely many $i \in I$. We record that $\mathcal{C}$ is stable under direct sums of finitely generated groups.

Proposition 6.5.4. *If $(H_n)_{n\geq 0}$ are finitely generated groups, then every minimal type III action of $\bigoplus_{n\geq 0} H_n$ factors through the projection to one H_m. In particular if all $H_n \in \mathcal{C}$, then $\bigoplus_{n\geq 0} H_n \in \mathcal{C}$.*

Proof. Denote $H = \bigoplus_{n\geq 0} H_n$, and consider $\varphi \in \mathrm{Rep}_{\mathrm{III}}(H)$. Suppose by contradiction that $\varphi(H_m)$ and $\varphi(\bigoplus_{n\neq m} H_n)$ are both non-trivial for some m. Then $\varphi(H_m)$ is a non-trivial normal subgroup of $\varphi(H)$, and since it is finitely generated, it has a minimal set. Then it is it either acts minimally, or it is a cyclic subgroup contained in the center of $\varphi(H)$, see [BMBRT24, Lemma 8.3.6]. The second possibility is excluded since φ is type III and has trivial centralizer. If $\varphi(H_m)$ is minimal, then $\varphi(\bigoplus_{n\neq m} H_n)$ is contained in the centralizer of a minimal action, hence it is abelian, hence contained in the center of $\varphi(H)$, also contradicting that φ is type III. The last assertion follows from Theorem 6.3.4. □

Recall that given groups H, K and an action $K \curvearrowright X$ on a set, the permutational wreath product $H \wr_X K$ is defined as the semi-direct product

$$\bigoplus_X H \rtimes K,$$

where K acts on $\bigoplus_X H$ by shifting coordinates. The wreath product $H \wr K$ is the permutational wreath product associated to the left-regular action of K on itself. The group $H \wr_X K$ is finitely generated provided H, K are finitely generated and the action $K \curvearrowright X$ has finitely many orbits. Hence Corollary 6.5.3 and Proposition 6.5.4 imply the following.

Corollary 6.5.5. *If $H, K \in \mathcal{C}$ are finitely generated, then $H \wr_X K \in \mathcal{C}$ for every action $K \curvearrowright X$ with finitely many orbits.*

6.5.2 Fundamental groups of graphs of groups

For more details on the material in this subsection see [Ser77].

For us, a *graph* Π consists in a set $V(\Pi)$ of vertices, a set $E(\Pi)$ of edges and two maps

$$e \in E(\Pi) \mapsto (o(e), t(e)) \in V(\Pi) \times V(\Pi) \quad \text{and} \quad e \in E(\Pi) \mapsto \overline{e} \in E(\Pi)$$

where $e \mapsto \overline{e}$ is a fixed-point-free involution and $o(\overline{e}) = t(e)$ for all $e \in E(\Pi)$. The vertex $o(e)$ (resp. $t(e)$) is the *origin* (resp. *terminus*) of e, and $\overline{e}$ is the *inverse edge* of e. A *graph of groups* $(\Pi, \mathcal{G})$ is the data of a connected graph Π and a collection $\mathcal{G}$ of groups $\{G_v\}_{v\in V(\Pi)}$ (the *vertex groups*) and $\{G_e\}_{e\in E(\Pi)}$ (the *edge groups*) with $G_e = G_{\overline{e}}$ for all $e \in E(\Pi)$, equipped with injective morphisms $\iota_{e,o(e)} \colon G_e \to G_{o(e)}$. The *fundamental group* $\pi_1(\Pi, \mathcal{G})$ of the graph of groups $(\Pi, \mathcal{G})$ is defined as follows: choose a connected subgraph $T \subseteq \Pi$ with no cycles and that contains all vertices of Π. Denote by t_e a symbol indexed by $e \in E(\Pi)$. Then $\pi_1(\Pi, \mathcal{G})$ is the quotient of the free product $\ast_{v\in V(\Pi)} G_v \ast \ast_{e\in E(\Pi\setminus T)} t_e$ by the relations

$$t_e \iota_{e,o(e)}(g) t_e^{-1} = \iota_{\overline{e},o(\overline{e})}(g) \quad \text{and} \quad t_e = t_{\overline{e}}^{-1}$$

for all $e \in E(\Pi \setminus T)$ and $g \in G_{t(e)}$, and $\iota_{e,o(e)}(g) = \iota_{e,t(e)}(g)$ for all $e \in E(T)$ and $g \in G_e$. HNN-extensions and amalgamated products correspond to the case when Π is a loop or a single edge, respectively. As in these special cases, the natural maps $G_v \to \pi_1(\Pi, \mathcal{G})$ and $G_e \to \pi_1(\Pi, \mathcal{G})$ are always injective [Ser77, Théorème 13], and one sees that $\pi_1(\Pi, \mathcal{G})$ is isomorphic to

$$\langle\langle G_v,\, v \in V(\Pi)\rangle\rangle \rtimes \langle t_e,\, e \in E(\Pi \setminus T)\rangle$$

where $\langle t_e,\, e \in E(\Pi \setminus T)\rangle \cong F_{b(\Pi)}$ is free of rank $b(\Pi)$, the first Betti number of the graph Π. The group $\pi_1(\Pi, \mathcal{G})$ does not depend, up to isomorphism, on the choice of T [Ser77, Proposition 20].

Bass-Serre theory [Ser77, §5] shows that presentations of a group G as a (finite) fundamental group of a graph of groups are in correspondence with (cocompact) actions of G on trees $\mathcal{T}$ by tree automorphisms with no inversions, that is, such that there is no $g \in G$ and $e \in E(\mathcal{T})$ with $g.e = \overline{e}$. One direction of this correspondence takes a graph of groups $(\Pi, \mathcal{G})$ and produces a canonically defined action of $\pi_1(\Pi, \mathcal{G})$ on a tree $\mathcal{T}_{\Pi,\mathcal{G}}$, called the *Bass-Serre tree* of $(\Pi, \mathcal{G})$, such that $\Pi = G \backslash \mathcal{T}_{\Pi,\mathcal{G}}$. The conjugates of the vertex groups G_v (resp. of the edge groups G_e) in $\pi_1(\Pi, \mathcal{G})$ are exactly the stabilizers of vertices (resp. edges) that project to v (resp. e) in Π, and the maps $\iota_{e,o(e)}$ are induced by the inclusion of edge stabilizers into vertex stabilizers. Moreover, the tree $\mathcal{T}_{\Pi,\mathcal{G}}$ is locally finite if and only if $\iota_{e,o(e)}(G_e)$ has finite index in $G_{o(e)}$ for all $e \in E(\Pi)$.

The following lemma is well known, and we include it for completeness.

Lemma 6.5.6. *Let $(\Pi, \mathcal{G})$ be a graph of groups where all inclusions $\iota_{e,o(e)}$, $e \in E(\Pi)$ have finite-index image in $G_{o(e)}$. Then any two stabilizers of vertices or edges in the Bass-Serre tree $\mathcal{T}_{\Pi,\mathcal{G}}$ are commensurate. In particular, any vertex group or edge group in $\pi_1(\Pi, \mathcal{G})$ is commensurated in $\pi_1(\Pi, \mathcal{G})$.*

If H is any vertex group or edge group, then $\mathrm{I}(H) = \langle\langle G_v,\, v \in V(\Pi)\rangle\rangle$, and thus $G/\mathrm{I}(H)$ is free of rank $b(\Pi)$.

Proof. Write $G = \pi_1(\Pi, \mathcal{G})$. The tree $\mathcal{T}_{\Pi,\mathcal{G}}$ is locally finite, so its balls for the natural path metric d are finite. If u, u' are vertices in $\mathcal{T}_{\Pi,\mathcal{G}}$, then $\mathrm{Stab}_G(u) \cap \mathrm{Stab}_G(u')$ has index at most $|\{w \in \mathcal{T}_{\Pi,\mathcal{G}} : d(u,w) = d(u,u')\}|$ inside $\mathrm{Stab}_G(u)$. The same is true if u or u' are edges in $\mathcal{T}_{\Pi,\mathcal{G}}$ since edge groups have finite index in vertex groups. If H is any vertex group or edge group, then $\mathrm{I}(H)$ is normal and contains all vertex groups (see Lemma 6.4.1), so contains $\langle\langle G_v,\, v \in V(\Pi)\rangle\rangle$, and we actually actually have equality since $G/\langle\langle G_v,\, v \in V(\Pi)\rangle\rangle = F_{b(\Pi)}$ is torsion-free. □

Corollary 6.5.7. *Suppose G is a finitely generated group that can be presented as the fundamental group $\pi_1(\Pi, \mathcal{G})$ of a finite graph of groups $(\Pi, \mathcal{G})$ where all the inclusions $\iota_{e,o(e)}$, $e \in E(\Pi)$ have finite-index image in $G_{o(e)}$ and every G_v, $v \in V(\Pi)$ is in $\mathcal{C}$. Then $G \in \mathcal{C}$ if and only if $b(\Pi)$ is at most 1.*

Proof. Since the non-abelian free group F_m is not in $\mathcal{C}$ for $m \geq 2$ and the class $\mathcal{C}$ is obviously closed under quotient, we have $b(\Pi) \leq 1$ if $G \in \mathcal{C}$. The converse holds by Theorem 6.5.2 and Lemma 6.5.6. □

Example 6.5.8. Recall that a group G is a *generalized Baumslag-Solitar group of rank $n \in \mathbb{N}_+$* if it is the fundamental group of a finite graph of groups where all vertex and edge groups are isomorphic to $\mathbb{Z}^n$. Corollary 6.5.7 characterizes exactly which generalized Baumslag-Solitar groups belong to $\mathcal{C}$.

The class of rank 2 generalized Baumslag-Solitar groups contains the examples of Leary-Minasyan of the first CAT(0) and non-virtually biautomatic groups [LM21], and the rank 1 generalized Baumslag-Solitar groups has been previously examined from geometric and algebraic perspectives as a natural generalization of the Baumslag-Solitar groups, see for instance [Why01, Lev07].

Proof of Corollary 6.1.3. Corollary 6.5.3 implies (i), Corollary 6.5.5 implies (ii), and Corollary 6.5.7 implies (iii) and (iv). □

6.6 Micro-supported and piecewise projective groups

This section proves Theorem M by showing a general result (Proposition 6.6.5) for micro-supported groups acting minimally on $\mathbb{R}$ and then specializing to piecewise projective groups. In this section, whenever given a group $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ we will call this distinguished action on $\mathbb{R}$ the *standard action* of G and denote it by $(g, x) \in G \times \mathbb{R} \mapsto g.x$. For simplicity of exposition in the proofs we will use Theorem 6.3.4 for sequences of translations instead of sequences of homeomorphisms.

A subgroup $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ is said to be *micro-supported* if for every relatively compact interval $I \subseteq \mathbb{R}$ the subgroup

$$G_I = \{g \in G : g(x) = x \text{ for all } x \in \mathbb{R} \setminus I\}$$

is non-trivial. Notice that specifying a micro-supported group also implies fixing an embedding $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$.

Proposition 6.6.1 ([BMBRT24, Chapter 3])**.** *Let $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ be a countable group acting minimally on $\mathbb{R}$. Then G is micro-supported if and only if it contains an element of relatively compact support.*

In this case, G admits a maximal normal subgroup $[G_c, G_c]$ where G_c is the subgroup of compactly supported elements of G for the standard action.

One of the main theorems of [BMBRT24] says that any faithful minimal action of a micro-supported G that is not itself micro-supported is necessarily laminar. We need a more detailed version of this statement making explicit some conditions on the resulting lamination, and some more notation.

Definition 6.6.2. *Given a group $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$, we say that a (possibly unbounded) non-empty interval $I \subsetneq \mathbb{R}$ is G-*good *if, by denoting $\mathcal{O}_I = \{g.I : g \in G\}$, the following properties are verified:*

- *for every $I_1, I_2 \in \mathcal{O}_I$ with $I_1 \cap I_2 \neq \varnothing$, either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$, and*
- *for every $I_1, I_2 \in \mathcal{O}_I$ there exists $I_3 \in \mathcal{O}_I$ such that $I_3 \supseteq I_1 \cup I_2$.*

If $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$, we say that I generates a lamination for φ *if one of the following properties is verified.*

- *Either for every (equivalently, for some) $K \in \mathcal{O}_I$ the support $\mathrm{supp}_\varphi([G_K, G_K])$ has bounded connected components, or*

- *for every (equivalently, for some) $K \in \mathcal{O}_I$ the group $\varphi([G_K, G_K])$ acts on $\mathbb{R}$ with no minimal set.*

If I is a G-good interval generating a lamination for $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$, the wandering intervals $\bigcup_{K \in \mathcal{O}_I} \mathcal{W}_\varphi([G_K, G_K])$ define a φ-invariant covering prelamination [BMBRT24, Proposition 8.3.4].

Lemma 6.6.3 ([BMBRT24, Section 9.1])**.** *Let $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ be a micro-supported finitely generated group acting minimally on $\mathbb{R}$, and let $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)$ be a faithful action that is not conjugate to the standard action. Then there exists a G-good interval $I \subsetneq \mathbb{R}$ such that I generates a lamination for φ.*

Lemma 6.6.4. *Let $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ be a finitely generated group acting irreducibly on $\mathbb{R}$, let $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G) \cap \mathrm{Harm}(G)$ and let $(t_n)_{n \geq 0} \subseteq \mathbb{R}$ such that $\lim_{n \to \infty} \Psi^{t_n}(\varphi) = \varphi$. If there is a G-good interval that generates a lamination for φ, then $\lim_{n \to \infty} t_n = 0$.*

Proof. Fix $S \subseteq G$ a finite symmetric generating set. Let

$$\delta_S = \sup\{|\varphi(s).y - y| : s \in S,\, y \in \mathbb{R}\},$$

which is finite since $\varphi \in \mathrm{Harm}(G)$ (see Remark 6.2.7). Let $I \subsetneq \mathbb{R}$ be a G-good interval such that

$$\mathcal{L}_\varphi = \bigcup_{K \in \mathcal{O}_I} \mathcal{W}_\varphi([G_K, G_K])$$

is a covering prelamination. Since I is G-good, the set $\bigcup_{K \in \mathcal{O}_I} [G_K, G_K]$ is a normal subgroup of G, whose image through φ acts irreducibly on $\mathbb{R}$. Hence there exists $\widetilde{K} \in \mathcal{O}_I$, $x \in \widetilde{K}$ and $g \in [G_{\widetilde{K}}, G_{\widetilde{K}}]$ such that $|\varphi(g).x - x| > \delta_S$. By enlarging $\widetilde{K}$ if necessary, we may assume that $s.\widetilde{K} \cap \widetilde{K} \neq \varnothing$ for all $s \in S$.

Claim 1. *For any $J \in \mathcal{L}_\varphi$ there exists $K \in \mathcal{O}_I$ with $K \supseteq \widetilde{K}$ such that $\mathcal{W}_\varphi([G_K, G_K])$ contains an interval $\hat{J} \supseteq J$.*

Proof of the claim. Suppose first that the $\varphi([G_K, G_K])$, $K \in \mathcal{O}_I$ act without a minimal set. Then $\mathcal{W}_\varphi([G_{\widetilde{K}}, G_{\widetilde{K}}])$ is a covering prelamination for $\varphi([G_{\widetilde{K}}, G_{\widetilde{K}}])$ by Lemma 6.4.5, so we may take $K = \widetilde{K}$.

If all the supports $\mathrm{supp}_\varphi([G_K, G_K])$, $K \in \mathcal{O}_I$ have a bounded connected component, since $\mathcal{L}_\varphi$ is a covering prelamination we may find $K' \in \mathcal{O}_I$ such that there is a connected component J' of $\varphi([G_{K'}, G_{K'}])$ containing J. Since I is G-good, there is a $K \in \mathcal{O}_I$ with $K \supseteq K' \cup \widetilde{K}$. If $\hat{J}$ is the connected component of $\mathrm{supp}_\varphi([G_K, G_K])$ containing J' we are done. □

Now fix $\widetilde{J} \in \mathcal{L}_\varphi$ such that $\varphi(s).\widetilde{J} \cap \widetilde{J} \neq \varnothing$ for all $s \in S$ and $x, \varphi(g).x \in \widetilde{J}$, so $\varphi(g).\widetilde{J} = \widetilde{J}$. Consider an arbitrary $J \in \mathcal{L}_\varphi$ containing $\widetilde{J}$ and take intervals $K \in \mathcal{O}_I$, $\hat{J} \supseteq J$ supplied by Claim 1 applied to J. We say that a (finite or infinite) sequence $(s_n)_{n \geq 0} \subseteq S$ is *K-admissible* if $(s_n \cdots s_1).K \supseteq K$ for all $n \geq 0$. Notice that when $(s_k \cdots s_1).K \supseteq K$, then the interval $\varphi(s_k \cdots s_1).\hat{J}$ is wandering for $\varphi([G_K, G_K])$ and is thus fixed by $\varphi(g)$ whenever $\varphi(s_k \cdots s_1).\hat{J} \supseteq \widetilde{J}$.

Claim 2. *Either there exists a K-admissible sequence $(s_n)_{n\geq 0}$ such that the collection of leaves $\{\varphi(s_n\cdots s_1).\hat{J}\}_{n\geq 0}$ is unbounded in $\mathcal{L}_\varphi$, or there exists a K-admissible sequence $(s_n)_{n\geq 0}$ and $m\in\mathbb{N}$ such that $\varphi(s_m\cdots s_1).\hat{J}\subsetneq\widetilde{J}$.*

Proof. Towards a contradiction, suppose that for every K-admissible sequence $(s_n)_{n\geq 0}$ the intervals $\varphi(s_n\cdots s_1).\hat{J}$ remain bounded and never lie inside $\widetilde{J}$. Let $J_M\in\mathcal{L}_\varphi$ such that $\varphi(s_n\cdots s_1).\hat{J}\subseteq J_M$ for all $n\in\mathbb{N}$. By irreducibility of φ, we can find a finite sequence $d_1,\ldots,d_k\in S$ such that $\varphi(d_k\cdots d_1).J_M\subsetneq\widetilde{J}$ and $\varphi(d_j\cdots d_1).J_M\supseteq\widetilde{J}$ for all $0\leq j<k$. By irreducibility of the standard action of G, we may find a finite sequence $e_1,\ldots,e_l\in S$ such that the sequence $e_1,\ldots,e_l,d_1,\ldots,d_k$ is K-admissible. But

$$\varphi(d_k\cdots d_1e_l\cdots e_1).\hat{J}\subseteq\varphi(d_k\cdots d_1).J_M\subsetneq\widetilde{J},$$

a contradiction. □

If for some $J\supseteq\widetilde{J}$ the intervals $K,\hat{J}$ verify the first option of the previous claim, then we conclude that $\mathrm{Fix}(\varphi(g))$ is δ_S-dense outside of $\hat{J}$. If for every $J\supseteq\widetilde{J}$ the intervals $K,\hat{J}$ verify the second option, then again we conclude that $\mathrm{Fix}(\varphi(g))$ is δ_S-dense outside of $\widetilde{J}$. In any case, t_n-x cannot exit a bounded interval around $x,\varphi(g).x$ since $|\Psi^{t_n}(\varphi)(g).x-x|>\delta_S$ for n large enough. Thus $(t_n)_{n\geq 0}$ is bounded and $\lim_{n\to\infty}t_n=0$ again by Theorem 6.3.4. □

Proposition 6.6.5. *Let $G\subseteq\mathrm{Homeo}_0(\mathbb{R})$ be a micro-supported finitely generated group acting minimally on $\mathbb{R}$. Then $G\in\mathcal{C}$ if and only if $G/[G_c,G_c]\in\mathcal{C}$.*

Proof. One implication is clear, so suppose that $G/[G_c,G_c]\in\mathcal{C}$. Consider $\varphi\in\mathrm{Rep}_{\mathrm{III}}(G)\cap\mathrm{Harm}(G)$ and $(t_n)_{n\geq 0}\subseteq\mathbb{R}$ such that $\lim_{n\to\infty}\Psi^{t_n}(\varphi)=\varphi$. Recall that the action φ is either faithful or factors through $G/[G_c,G_c]$, so we will prove the proposition by assuming that φ is faithful and showing that $\lim_{n\to\infty}t_n=0$ in this case.

Suppose first that φ is micro-supported. Fix an open bounded interval $I\subseteq\mathbb{R}$, take a non-trivial element $g\in G$ with $\mathrm{supp}_\varphi(g)\subseteq I$ and let $x\in I$ such that $\varphi(g).x\neq x$. But if $x-t_n\notin I$ we have $\Psi^{t_n}(\varphi)(g).x=x$, so $x-t_n\in I$ from some $n\in\mathbb{N}$ onwards and hence $(t_n)_{n\geq 0}$ is bounded. By Theorem 6.3.4 we conclude that $\lim_{n\to\infty}t_n=0$.

Now suppose that φ is not micro-supported, so φ is not semiconjugate to the standard action. By Lemma 6.6.3 there is a G-good interval that generates a lamination for φ, and Lemma 6.6.4 shows that $\lim_{n\to\infty}t_n=0$. We conclude that $G\in\mathcal{C}$ by Theorem 6.3.4. □

Example 6.6.6. Proposition 6.6.5 is already enough to supply many new examples of elementary amenable subgroups of $\mathrm{Homeo}_0(\mathbb{R})$ that are in $\mathcal{C}$ and are not virtually solvable, for instance the Brin-Navas group B defined in [Nav04, Bri05]. The group B can be defined as follows: fix a non-empty open bounded interval $(x,y)\subseteq\mathbb{R}$ and $f\in\mathrm{Homeo}_0(\mathbb{R})$ such that the only fixed point of f is a single point in (x,y) and such that $f(x)<x<y<f(y)$. Choose a non-decreasing homeomorphism $w_0\in\mathrm{Homeo}_0(\mathbb{R})$ whose support is (x,y) and such that $w_0(f^{-1}(x))=f^{-1}(y)$. Then B is the group generated by f,w_0, and its defining action on $\mathbb{R}$ is minimal and micro-supported. It is clear that these conditions can be met by piecewise affine homeomorphisms f,w_0.

The conditions on w_0 imply that w_0 and $w_1 = fw_0f^{-1}$ are such that w_0 and $w_1w_0w_1^{-1}$ have disjoint support, so w_0, w_1 generate a wreath product $\mathbb{Z}\wr\mathbb{Z}$. Similarly, the subgroup generated by any finite subset of the $w_n = f^n w_0 f^{-n}, n \in \mathbb{Z}$ is an iterated wreath product $(\cdots((\mathbb{Z}\wr\mathbb{Z})\wr\mathbb{Z})\cdots)\wr\mathbb{Z}$. Thus B is non-virtually solvable and elementary amenable. See [BBM21] for more examples of this kind.

Remark. Suppose $G \subseteq \mathrm{Homeo}_0(\mathbb{R})$ is a group acting minimally on $\mathbb{R}$ that admits a generating set $\{s_1, s_2, \dots, s_k\}$ where s_1 is supported on an interval $(-\infty, x)$, s_k is supported on an interval (y, ∞) and the $s_2, \dots, s_{k-1}$ have relatively compact support. Then G is micro-supported and writing $G/[G_c, G_c]$ as the extension (6.6.1) shows that $G/[G_c, G_c]$ is solvable, so by Proposition 6.6.5 we have $G \in \mathcal{C}$. This applies to the family of pre-chain groups studied in [KKL19], who were already known to lie in $\mathcal{C}$ from [BMBRT24, Section 16.3.3].

Lemma 6.6.7 ([BMBRT24, Lemma 9.1.2])**.** *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ and $H_1, H_2 \subseteq G$ two commuting subgroups such that $\varphi(H_1)$ admits a minimal set Λ. Then either $[H_1, H_1]$ or $[H_2, H_2]$ fixes Λ pointwise.*

Proposition 6.6.8. *Let $G \subseteq \mathrm{PProj}_0(\mathbb{R})$ be a finitely generated group acting irreducibly on $\mathbb{R}$. Then $G \in \mathcal{C}$.*

Proof. Let $\Lambda \subseteq \mathbb{R}$ be a minimal set for the standard action of G. Let $N \subseteq G$ the kernel of the action on Λ, which is composed of compactly supported homeomorphisms.

Claim 1. *The group $G/[N, N]$ belongs to $\mathcal{C}$.*

Proof of the claim. Set $Q = G/N$. We first prove that $Q \in \mathcal{C}$. Notice that Q is either abelian (and the conclusion follows) or admits a minimal faithful action on $\mathbb{R}$ which is a continuous factor of the standard action of G. In this case, the groups of germs of Q at $-\infty, \infty$ are isomorphic to those of G and both are 2-step solvable. Unless Q is 2-step solvable (and the conclusion follows from Proposition 6.5.2), there exists $g_1, g_2, g_3, g_4 \in Q$ such that $[[g_1, g_2], [g_3, g_4]]$ is a non-trivial homeomorphism of relatively compact support. By Proposition 6.6.1 this implies that Q is micro-supported, and Proposition 6.6.5 shows that $Q \in \mathcal{C}$ if and only if $Q/[Q_c, Q_c]$. But $Q/[Q_c, Q_c]$ is finitely generated and can be written as the extension

$$1 \longrightarrow Q_c/[Q_c, Q_c] \longrightarrow Q/[Q_c, Q_c] \longrightarrow Q/Q_c \longrightarrow 1 \tag{6.6.1}$$

where $Q_c/[Q_c, Q_c]$ is abelian and Q/Q_c solvable since it is contained in the product of the groups of germs of Q at $-\infty, \infty$, which are subgroups of $\mathrm{Aff}(\mathbb{R})$. We conclude that $Q/[Q_c, Q_c] \in \mathcal{C}$ by Proposition 6.5.2. Hence $Q \in \mathcal{C}$ in any case.

By writing the finitely generated group $G/[N, N]$ as an extension of Q by the abelian group $N/[N, N]$ we conclude that $G/[N, N] \in \mathcal{C}$ by Proposition 6.5.2. □

Now let $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G) \cap \mathrm{Harm}(G)$ and $(t_n)_{n \geq 0} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} \Psi^{t_n}(\varphi) = \varphi$. If $\varphi([N, N])$ is trivial, then $\lim_{n\to\infty} t_n = 0$ because $G/[N, N] \in \mathcal{C}$. Since N is composed of compactly supported homeomorphisms, we may assume that the subgroups $\varphi([N_{(x,\infty)}, N_{(x,\infty)}])$ and $\varphi([N_{(-\infty,x)}, N_{(-\infty,x)}])$ are non-trivial for every $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$.

Claim 2. *One of the G-good intervals $(x,\infty),(-\infty,x)$ generates a lamination for φ.*

Proof of the claim. Suppose first that $\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ has a global fixed point. If the support $\operatorname{supp}_\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ has a bounded connected component C, then the claim follows. If not, then one of the connected components of $\operatorname{supp}_\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ is an interval of the form $(h(x),\infty)$ or $(-\infty,h(x))$, and we may assume that it is of the form $(h(x),\infty)$ (the other case is analogous). Define a map h from $\mathrm{Orb}_{\mathrm{st}}(x)$, the orbit of x under the standard action of G, to $\mathbb{R}$ by setting

$$y \in \mathrm{Orb}_{\mathrm{st}}(x) \mapsto \inf \operatorname{supp}_\varphi([G_{(y,\infty)},G_{(y,\infty)}]).$$

The map h intertwines the standard action of G with φ and is monotone. Up to replacing it by $x \mapsto -h(x)$ it extends to a semiconjugacy between φ and the standard action of G. Since φ is minimal, h is continuous. Hence the standard action of G does not have a discrete orbit and φ is conjugate to the minimal action of G/N which is a factor of the standard action of G. This is a contradiction, since $\varphi([N,N])$ is non-trivial.

By repeating the same argument as the previous paragraph with $\varphi([G_{(-\infty,x)},G_{(-\infty,x)}])$, we may suppose then that the groups $\varphi([G_{(x,\infty)},G_{(x,\infty)}]),\varphi([G_{(-\infty,x)},G_{(-\infty,x)}])$ act irreducibly. Suppose first that $\varphi(G_{(x,\infty)})$ admits a minimal set $\Lambda \subseteq \mathbb{R}$. Since the subgroups $\varphi(G_{(x,\infty)})$ and $\varphi(G_{(-\infty,x)})$ commute, Lemma 6.6.7 shows that one of $\varphi([G_{(-\infty,x)},G_{(-\infty,x)}]),\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ must fix Λ pointwise, contradicting the irreducibility of their action.

Suppose instead that $\varphi(G_{(x,\infty)})$ admits no minimal set, so its action on $\mathbb{R}$ is irreducible and not cocompact. Thus the action of $\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ is irreducible and not cocompact either. We deduce that $\varphi([G_{(x,\infty)},G_{(x,\infty)}])$ admits no minimal set, which implies the claim in this case. □

By Lemma 6.6.4 we conclude that $\lim_{n\to\infty} t_n = 0$. Hence $G \in \mathcal{C}$ by Theorem 6.3.4. □

Corollary 6.6.9 (Theorem M)**.** *Let $G \subseteq \mathrm{PProj}_0(\mathbb{R})$ be a finitely generated group. Then $G \in \mathcal{C}$.*

Proof. To prove the general case when the standard action of G is not necessarily irreducible, we argue by induction on the number c of the connected components of its support $\operatorname{supp}_{\mathrm{st}}(G)$. This set is finite, since the endpoints of connected components of $\operatorname{supp}_{\mathrm{st}}(G)$ are included in the set $\bigcup_{s\in S}\partial\mathrm{Fix}(s)$ where $S \subseteq G$ is a finite generating set.

If $c=1$, then Lemma 6.6.8 shows $G \in \mathcal{C}$ because the group of piecewise projective homeomorphisms of any open interval is isomorphic to $\mathrm{PProj}_0(\mathbb{R})$. Assume $c \geq 2$, denote by $I \subseteq \mathbb{R}$ the leftmost connected component of $\operatorname{supp}_{\mathrm{st}}(G)$ and define J as the smallest open interval containing all connected components of $\operatorname{supp}_{\mathrm{st}}(G)$ except for I. Write N_I for the kernel of the action of G on I and G_I for the image of the restriction of G to I, and likewise with N_J, G_J.

Consider $\varphi \in \mathrm{Rep}_{\mathrm{III}}(G)\cap\mathrm{Harm}(G)$ and $(t_n)_{n\geq 0} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty}\Psi^{t_n}(\varphi)=\varphi$. If $\varphi(N_I)$ or $\varphi(N_J)$ have global fixed points then they are trivial and we conclude that $\lim_{n\to\infty} t_n = 0$ since $G_I = G/N_I$ and $G_J = G/N_J$ are in $\mathcal{C}$ by the inductive hypothesis. Thus we may assume that $\varphi(N_I)$ and $\varphi(N_J)$ act irreducibly on $\mathbb{R}$. Notice that $\varphi(N_I)$ cannot be of type III, since it has a non-trivial centralizer $\varphi(N_J)$. We conclude that $\lim_{n\to\infty} t_n = 0$ by Lemma 6.5.1. Again $G \in \mathcal{C}$ by Theorem 6.3.4, finishing the proof. $\square$

6.7 Complexity of the semiconjugacy relation

This section is devoted to the proof of Theorem N. Subsection 6.7.1 provides the necessary background on preorders on groups and some preliminary lemmas. Subsection 6.7.2 proves the theorem by reducing semiconjugacy among cocompact actions of a group G to conjugacy on a (Borel) subset of $\mathrm{Rep}_{\min}(G) \cup \mathrm{Rep}_{\mathrm{cyc}}(G)$ obtained as the dynamical realization of a certain space of preorders on G. The use of this space of preorders also makes apparent the essential countability of all the equivalence relations involved.

In this section G is always a countable group. Recall that an irreducible action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ is *cocompact* if there is a compact subset of $\mathbb{R}$ intersecting every φ-orbit. This condition is equivalent to the existence of a minimal set for φ, see e.g. [BMBRT24, Lemma 2.1.11]. This in turn is equivalent to the fact that φ is semiconjugate to a minimal or cyclic action in $\mathrm{Rep}_{\mathrm{irr}}(G)$, which is unique up to conjugacy. Such an action will be called a *canonical model* for φ. The space of all cocompact actions of G on $\mathbb{R}$ is denoted $\mathrm{Rep}_{\mathrm{cc}}(G)$.

6.7.1 Preorders and dynamical realizations

A *left-preorder* on G is a total, reflexive and transitive binary relation $\preceq$ on G that is also left-invariant, that is, such that $h \preceq k$ implies $gh \preceq gk$ for all $g, h, k \in G$. A left-preorder $\preceq$ is determined by the data of its *positive cone* $P_{\preceq} = \{g \in G : g \succeq e_G\}$ and its *residue subgroup* $[1]_{\preceq} = \{g \in G : e_G \preceq g \preceq e_G\}$. We will use the word *preorder* to designate a left-preorder that is not the *trivial preorder* $\preceq_{\mathrm{tr}}$ with $[1]_{\preceq_{\mathrm{tr}}} = G$. Denote by $\mathrm{LPO}(G)$ the set of preorders on G with the topology inherited from $\{\preceq, \npreceq\}^{G\times G}$, which becomes a locally compact and totally disconnected Polish space with the induced topology (see [DR19, Theorem 2.30]).

There is a dictionary between preorders and irreducible actions that originates in the proof of the fact that a countable group is left-orderable (that is, admits a preorder where the residue subgroup is trivial) if and only if it embeds into $\mathrm{Homeo}_0(\mathbb{R})$, see [Ghy01, Theorem 6.8]. The interplay between both viewpoints has been exploited to obtain results on the orders of a group [Nav10] and to study groups acting faithfully on the line [Mor06]. We refer to [DNR16] for a complete treatment of the subject.

We introduce two operations to realize this translation. Given an action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ denote by $\preceq_\varphi \in \mathrm{LPO}(G)$ the preorder on G where, for any $g, h \in G$, $g \preceq_\varphi h$ if and only $\varphi(g).0 \leq \varphi(h).0$. Given a preorder $\preceq \in \mathrm{LPO}(G)$, following [Ghy01, Theorem 6.8] define the *dynamical realization* $\phi_{\preceq} \in \mathrm{Rep}_{\mathrm{irr}}(G)$ of $\preceq$ as the action of G constructed as follows: fix once and for all a numbering $(g_i)_{i\geq 0}$ of G with $g_0 = e_G$. Define an order-preserving embedding $\iota_{\preceq} \colon G \to \mathbb{R}$ inductively by setting $\iota_{\preceq}(g_0) = 0$ and, given the values $\iota_{\preceq}(g_0), \ldots, \iota_{\preceq}(g_n)$, set

$$\iota_{\preceq}(g_{n+1}) = \begin{cases} \max\{\iota_{\preceq}(g_0), \ldots, \iota_{\preceq}(g_n)\} + 1 & \text{if } g_{n+1} \succ g_0, \ldots, g_n \\ \min\{\iota_{\preceq}(g_0), \ldots, \iota_{\preceq}(g_n)\} - 1 & \text{if } g_{n+1} \prec g_0, \ldots, g_n \\ (\iota_{\preceq}(g_i) + \iota_{\preceq}(g_j))/2 & \text{otherwise,} \end{cases}$$

where

$$g_j = \max\{g_m : 0 \leq m \leq n \text{ and } g_m \preceq g_{n+1}\} \text{ and } g_i = \min\{g_m : 0 \leq m \leq n \text{ and } g_m \succeq g_{n+1}\}.$$

The group G acts on $\iota_{\preceq}(G)$ by $g.\iota_{\preceq}(h) = \iota_{\preceq}(gh)$, and this action extends continuously to $\overline{\iota_{\preceq}(G)}$. We define $\phi_{\preceq}$ by extending the action of G on $\overline{\iota_{\preceq}(G)}$ affinely on its complement in $\mathbb{R}$.

Given a preorder $\preceq$, a subgroup $H \subseteq G$ is said to be *$\preceq$-convex* if whenever $h \in H$ and $g \in G$ are such that $e_G \preceq g \preceq h$, we have $g \in H$. The set of all $\preceq$-convex subgroups is totally ordered, and we denote $H_{\preceq}$ as the union of all proper $\preceq$-convex subgroups. When $H_{\preceq} \neq G$ (which is always the case when G is finitely generated, see [DNR16, Example 2.1.2]), we define $\preceq_*$ as the preorder on G where $P_{\preceq_*} = P_{\preceq} \setminus H_{\preceq}$ and $[1]_{\preceq_*} = H_{\preceq}$. If $H_{\preceq} = G$, we set $\preceq_* = \preceq_{\mathrm{tr}}$. When non-trivial, the preorder $\preceq_*$ is called the *minimal model* of $\preceq$ and plays the same role as the canonical model of a cocompact action $\varphi \in \mathrm{Rep}_{\mathrm{cc}}(G)$. The situation $H_{\preceq} = G$ is the order-theoretic analogue of the non-existence of a minimal set for an action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$.

Proposition 6.7.1 ([BMBRT24, Section 14.3.2])**.** *Let $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G)$ and $\preceq \in \mathrm{LPO}(G)$.*

i. The action φ is semiconjugate to the dynamical realization of $\preceq_{\varphi}$.

ii. The dynamical realization of $\preceq$ is cocompact if and only if $\preceq$ has a maximal convex subgroup $H_{\preceq} \neq G$. In that case, the dynamical realization of $\preceq_*$ is a canonical model for the dynamical realization of $\preceq$.

iii. The action φ is cocompact if and only if $\preceq_{\varphi}$ has a minimal model.

The term *dynamical realization* is usually used in the literature to denote any conjugate of $\phi_{\preceq}$ (see [DNR16, Section 1.1.3] for instance). The only property that we will need from this definition of $\phi_{\preceq}$, aside from the fact that $\iota_{\preceq}$ is explicit, is the following.

Lemma 6.7.2. *If $x \in \mathbb{Z}[1/2]$ and $\preceq \in \mathrm{LPO}(G)$, then either $x \in \iota_{\preceq}(G)$ or $x \in \mathbb{R} \setminus \overline{\iota_{\preceq}(G)}$.*

Proof. Say that $y \in \mathbb{Z}[1/2]$ has *height* $n \in \mathbb{N}$ if it can be written as $y = k/2^n$ for some $k \in \mathbb{Z}$. Define $l_n(y)$, the *left neighbor of height* n of y, as the largest $z \in \mathbb{Z}[1/2]$ with height n such that $z \leq y$. Define the right neighbor $r_n(y)$ of height n of y similarly.

Claim. *If $x \in \iota_{\preceq}(G)$ has height n, then $l_k(x)$ and $r_k(x)$ belong to $\iota_{\preceq}(G)$ for every $0 \leq k < n$.*

Proof of the claim. We argue that this holds true by induction at every stage of the construction of $\iota_{\preceq}(G)$. We may assume that x does not have height k for any $0 \leq k < n$, since $l_j(x) = r_j(x) = x$ whenever x has height j. Thus if $x \in \mathbb{Z}$ then $n = 0$ and the statement follows, so we may assume that $n > 0$ and that x was added as a midpoint of elements $y_1, y_2 \in \iota_{\preceq}(G)$ with $y_1 < x < y_2$. Then $l_k(x) = l_k(y_1)$, $r_k(x) = r_k(y_2)$ for every $0 \leq k < n$, and $l_k(y_1), r_k(y_2) \in \iota_{\preceq}(G)$ by the inductive hypothesis. □

Now suppose that $x \in \mathbb{Z}[1/2]$ belongs to $\overline{\iota_{\preceq}(G)}$. Write $x = k/2^j$ for some $k \in \mathbb{Z}, j \in \mathbb{N}$. Let $(x_n)_{n\geq 0} \subseteq \iota_{\preceq}(G)$ be a sequence converging to x, all of whose elements may be assumed to be different from x. Consider $x_n \in ((k-1)/2^j, (k+1)/2^j)$ such that x_n does not have height j, so either $r_j(x_n) = x$ or $l_j(x_n) = x$. Hence $x \in \iota_{\preceq}(G)$ by the previous claim. □

By identifying a subgroup of G with its indicator function, the set $\mathrm{Sub}(G)$ of all subgroups of G becomes a subset of $\{0,1\}^G$ and inherits a compact metrizable topology [Cha50]. Its Borel structure is generated by the sets $\{H \in \mathrm{Sub}(G) : g \in H\}$ where $g \in G$. A subgroup $H \subseteq G$ is *proper* if $H \neq G$.

We now prove that the function assigning to a preorder its maximal convex subgroup is Borel.

Lemma 6.7.3. *The map* $\preceq \in \mathrm{LPO}(G) \mapsto H_{\preceq} \in \mathrm{Sub}(G)$ *is Borel.*

Proof. Fix $g \in G$ and consider the Borel set

$$B_g = \{(\preceq, H) \in \mathrm{LPO}(G) \times \mathrm{Sub}(G) : H \text{ is proper, } \preceq\text{-convex and contains } g\}.$$

Let $\pi\colon \mathrm{LPO}(G) \times \mathrm{Sub}(G) \to \mathrm{LPO}(G)$ be the projection onto the first coordinate, and consider a section $S = B_g \cap \pi^{-1}(\preceq)$ for some fixed preorder $\preceq$. Then S is homeomorphic to

$$\{H \in \mathrm{Sub}(G) : H \text{ is proper, } \preceq\text{-convex and contains } g\},$$

which can be written as

$$\bigcup_{u\in G} \bigcap_{v \succeq e_G} \left(\{H : u \notin H \text{ and } v, g \in H\} \cup \{H : u \notin H,\ g \in H, \text{ and } h \preceq v \text{ for all } h \in H\}\right). \quad (6.7.1)$$

Indeed, a subgroup $H \subseteq G$ is proper and $\preceq$-convex if and only if there exists $u \in G \setminus H$, and for every $v \succeq e_G$ either $v \in H$ or $h \preceq v$ for all $h \in H$. The expression (6.7.1) shows that S is K_σ (that is, a countable union of compact sets), so Theorem 6.2.4 implies that $\pi(B_g)$ is Borel and that there is a Borel map $\zeta_g\colon \mathrm{LPO}(G) \to \mathrm{Sub}(G)$ such that $(\preceq, \zeta_g(\preceq)) \in B_g$ if $\preceq \in \pi(B_g)$ and $\zeta_g(\preceq) = \{e_G\}$ if not.

The condition $\preceq \in \pi(B_g)$ is equivalent to the existence of a proper, $\preceq$-convex subgroup H containing g. Thus we may write $H_{\preceq} = \bigcup_{g\in G} \zeta_g(\preceq)$ for every $\preceq \in \mathrm{LPO}(G)$. We conclude that $\preceq \mapsto H_{\preceq}$ is Borel too. □

6.7.2 Proof of Theorem N

Denote by $\mathrm{LPO}_{\mathrm{can}}(G)$ the set of preorders $\preceq$ on G such that $H_{\preceq} = [1]_{\preceq}$. From Lemma 6.7.3 it is clear that $\mathrm{LPO}_{\mathrm{can}}(G) \subseteq \mathrm{LPO}(G)$ is Borel. To study semiconjugacy among cocompact actions (to show its essential countability, specifically) we introduce the equivalence relation $\mathcal{R}$ on $\mathrm{LPO}_{\mathrm{can}}(G)$ by declaring $\preceq \mathcal{R} \preceq'$ if the dynamical realizations $\iota_{\preceq}, \iota_{\preceq'} \in \mathrm{Rep}_{\min}(G) \cup \mathrm{Rep}_{\mathrm{cyc}}(G)$ are conjugate.

Proposition 6.7.4. *The relation of semiconjugacy on* $\mathrm{Rep}_{\mathrm{cc}}(G)$ *is Borel reducible to* $\mathcal{R}$*, which is in turn Borel reducible to conjugacy on* $\mathrm{Rep}_{\min}(G) \cup \mathrm{Rep}_{\mathrm{cyc}}(G)$ *(and all the spaces involved are standard Borel).*

Proof. By writing

$$\mathrm{Rep}_{\mathrm{irr}}(G) = \bigcap_{n\in\mathbb{N}} \bigcup_{g,h\in G} \{\varphi\colon G \to \mathrm{Homeo}_0(\mathbb{R}) : \varphi(g).0 > n,\ \varphi(h).0 < -n\},$$

$$\begin{aligned}\mathrm{Rep}_{\mathrm{cc}}(G) &= \{\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G) : \text{there is } n \geq 1 \text{ such that } \varphi(G).[-n,n] \text{ covers } \mathbb{R}\} \\ &= \bigcup_{n\geq 1} \bigcap_{k\in\mathbb{Z}} \bigcup_{\substack{F\subseteq G \\ \text{finite}}} \{\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G) : \varphi(F).[-n,n] \supseteq [k,k+1]\}\end{aligned}$$

and

$$\mathrm{Rep}_{\mathrm{cyc}}(G) = \bigcup_{h\in G} \bigcap_{g\in G} \bigcup_{n\in\mathbb{Z}} \{\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G) : \varphi(g) = \varphi(h)^n\}$$

we see that $\mathrm{Rep}_{\mathrm{irr}}(G)$ is G_δ in $\mathrm{Homeo}_0(\mathbb{R})^G$ and that $\mathrm{Rep}_{\mathrm{cc}}(G), \mathrm{Rep}_{\mathrm{cyc}}(G) \subseteq \mathrm{Rep}_{\mathrm{irr}}(G)$ are Borel.

Consider the maps

$$\eta_1\colon \varphi \in \mathrm{Rep}_{\mathrm{cc}}(G) \mapsto (\preceq_\varphi)_* \in \mathrm{LPO}_{\mathrm{can}}(G)$$

and

$$\eta_2\colon \ \preceq \in \mathrm{LPO}_{\mathrm{can}}(G) \mapsto \phi_\preceq \in \mathrm{Rep}_{\min}(G) \cup \mathrm{Rep}_{\mathrm{cyc}}(G).$$

Proposition 6.7.1, (iii) (resp. (ii)) implies that η_1 (resp. η_2) is well defined.

Claim 1. *η_1 is a Borel map.*

Proof of the claim. Write η_1 as the composition of

$$\varphi \in \mathrm{Rep}_{\mathrm{cc}}(G) \mapsto \preceq_\varphi \in \mathrm{LPO}(G) \quad \text{and} \quad \preceq \in \mathrm{LPO}(G) \mapsto \preceq_* \in \mathrm{LPO}(G) \cup \{\preceq_{\mathrm{tr}}\}.$$

The first map is Borel, since

$$\{\varphi \in \mathrm{Rep}_{\mathrm{cc}}(G) : e_G \preceq_\varphi g\} = \{\varphi \in \mathrm{Rep}_{\mathrm{cc}}(G) : \varphi(g).0 \geq 0\}$$

for any $g \in G$. To see that the second map is Borel, notice that for any $g \in G$ we have

$$\{\preceq \in \mathrm{LPO}(G) : e_G \preceq_* g\} = \{\preceq \in \mathrm{LPO}(G) : e_G \preceq g \text{ or } g \in H_\preceq\}$$

where $H_\preceq \subseteq G$ is the maximal convex subgroup of $\preceq$. Since $\preceq \mapsto H_\preceq$ is Borel by Lemma 6.7.3 we conclude that η_1 is also Borel. □

Claim 2. *η_2 is a Borel map.*

Proof of the claim. Let $x \in \mathbb{R}$ and fix $g \in G$. We want to prove that the set

$$\{\preceq \in \mathrm{LPO}(G) : \phi_{\preceq}(g).x > x\}$$

is Borel, and by density of $\mathbb{Z}[1/2]$ in $\mathbb{R}$ we may assume that $x \in \mathbb{Z}[1/2]$. By Lemma 6.7.2, we have that for every $\preceq \in \mathrm{LPO}(G)$ either $x \in \iota_{\preceq}(G)$ or $x \in \mathbb{R} \setminus \overline{\iota_{\preceq}(G)}$. Thus

$$\begin{aligned}\{\preceq \in \mathrm{LPO}(G) : \phi_{\preceq}(g).x > x\} = &\bigcup_{h\in G}\{\preceq : \iota_{\preceq}(h) = x \text{ and } gh \succ h\} \\ &\cup \bigcup_{h_1,h_2\in G} \{\preceq : \iota_{\preceq}(h_1) < x < \iota_{\preceq}(h_2),\, gh_1 \succ h_1 \\ &\text{where } h_1 = \max\{h \in G : h \prec h_2\}, h_2 = \min\{h \in G : h \succ h_1\}\}.\end{aligned} \tag{6.7.2}$$

The explicit construction of $\iota_{\preceq}$ from $\preceq$ implies that all the sets $\{\preceq : \iota_{\preceq}(h) = y\}$ where $h \in G$, $y \in \mathbb{Z}[1/2]$ are Borel, and we conclude that the set (6.7.2) is also Borel. □

Now let $\varphi \in \mathrm{Rep}_{\mathrm{cc}}(G)$. By definition we have that φ is semiconjugate to $\phi_{\preceq_{\varphi}}$, whose canonical model is $\phi_{(\preceq_{\varphi})_*} = \eta_2 \circ \eta_1(\varphi)$ by Proposition 6.7.1 (ii). This implies that η_1 is a reduction of semiconjugacy on $\mathrm{Rep}_{\mathrm{cc}}(G)$ to $\mathcal{R}$. By its very definition, $\mathcal{R}$ is reducible to conjugacy on $\mathrm{Rep}_{\min}(G) \cup \mathrm{Rep}_{\mathrm{cyc}}(G)$. □

Proposition 6.7.5. *The relation $\mathcal{R}$ is essentially countable.*

Proof. Let $\mathcal{F} \subseteq \mathrm{LPO}_{\mathrm{can}}(G)$ be the set of preorders $\preceq$ such that $\phi_{\preceq}$ is not of type I. Since the actions $\phi_{\preceq}$ are canonical models when $\preceq \in \mathrm{LPO}_{\mathrm{can}}(G)$, the set $\mathrm{LPO}_{\mathrm{can}}(G) \setminus \mathcal{F}$ can be written as

$$\{\preceq \in \mathrm{LPO}_{\mathrm{can}}(G) : \text{for every } g \in G, \mathrm{Fix}(\phi_{\preceq}(g)) = \varnothing \text{ or } \mathrm{Fix}(\phi_{\preceq}(g)) = \mathrm{id}_{\mathbb{R}}\}$$

so $\mathcal{F}$ is Borel by (the proof of) Proposition 6.7.4.

It is clear that $\mathcal{F}$ is $\mathcal{R}$-invariant.

Claim. *The relation $\mathcal{R}$ restricted to $\mathrm{LPO}_{\mathrm{can}}(G) \setminus \mathcal{F}$ is smooth.*

Proof of the claim. By the definition of $\mathcal{F}$, the relation $\mathcal{R}$ restricted to $\mathrm{LPO}_{\mathrm{can}}(G) \setminus \mathcal{F}$ Borel reduces to conjugacy in $\mathrm{Rep}_{\min}(G/[G,G]) \cup \mathrm{Rep}_{\mathrm{cyc}}(G/[G,G])$. Corollary 6.3.5 shows that conjugacy on $\mathrm{Rep}_{\min}(G/[G,G])$ is smooth. The same is true for conjugacy on $\mathrm{Rep}_{\mathrm{cyc}}(G)$ (and *a fortiori* on $\mathrm{Rep}_{\mathrm{cyc}}(G/[G,G])$): given $\varphi \in \mathrm{Rep}_{\mathrm{cyc}}(G)$ there exists a unique isomorphism between $\varphi(G) \subseteq \mathrm{Homeo}_0(\mathbb{R})$ and $\mathbb{Z}$ sending the generator f of $\varphi(G)$ with $f(0) > 0$ to $1 \in \mathbb{Z}$. We obtain a uniquely defined morphism $\pi_{\varphi} \colon G \to \mathbb{Z}$, and the assignment $\varphi \in \mathrm{Rep}_{\mathrm{cyc}}(G) \mapsto \pi_{\varphi} \in \mathrm{Hom}(G,\mathbb{Z})$ is a Borel reduction of the conjugacy relation on $\mathrm{Rep}_{\mathrm{cyc}}(G)$ to equality on the (standard Borel) space $\mathrm{Hom}(G,\mathbb{Z})$. □

Thus we may restrict our considerations to $\mathcal{F}$. We will show that the subset

$$T = \{\preceq \in \mathcal{F} : \text{there exists } g \in G \text{ such that } 0 \in \partial\mathrm{Fix}(\phi_{\preceq}(g))\}.$$

is a *complete countable Borel section* for $\mathcal{R}$ on $\mathcal{F}$, that is, T is Borel and intersects any $\mathcal{R}$-class $[\preceq]$ in $\mathcal{F}$ in a countable non-empty set. An application of the Lusin-Novikov theorem shows that $\mathcal{R}$ restricted to $\mathcal{F}$ is essentially countable (see [Gao09, Corollary 7.5.3], for instance).

To see that T is Borel, notice that a preorder $\preceq\in\mathcal{F}$ is in T if and only if there is $g\in[1]_\preceq$ such that either:

- there is $x\succ e_G$ such that for every $y\in G$ with $y\succ e_G$, there is $n\in\mathbb{N}$ with $g^nx\prec y$, or
- there is $x\prec e_G$ such that for every $y\in G$ with $y\prec e_G$, there is $n\in\mathbb{N}$ with $g^nx\succ y$.

This description readily implies that T is Borel in $\mathcal{F}$.

Now fix $[\preceq]$ an $\mathcal{R}$-class in $\mathcal{F}$. Since $\phi_\preceq$ is not of type I there is $g\in G$ such that $\mathrm{Fix}(\phi_\preceq(g))$ is not $\mathbb{R}$ nor $\varnothing$. Choose $x\in\partial\mathrm{Fix}(\phi_\preceq(g))$, define $\psi=\Psi^{-x}(\phi_\preceq)$ and let $\preceq'=\preceq_\psi$ be the left preorder induced by the $\psi(G)$-orbit of 0. By Proposition 6.7.1, (ii), we have $\preceq'\in\mathrm{LPO}_{\mathrm{can}}(G)$ so $\preceq'\in T$. Then $\phi_\preceq$, ψ and $\phi_{\preceq'}$ are all conjugate, hence $\preceq'\in T\cap[\preceq]$ and $T\cap[\preceq]$ is non-empty.

To see that $T\cap[\preceq]$ is at most countable, fo each $\preceq'\in T\cap[\preceq]$ consider $f\in\mathrm{Homeo}_0(\mathbb{R})$ such that $\phi_\preceq=f\circ\phi_{\preceq'}\circ f^{-1}$. Notice that $f(0)$ belongs to the countable set $\bigcup_{g\in G}\partial\mathrm{Fix}(\phi_\preceq(g))$ because $\preceq'\in T$. For any $g\in G$ we have the equivalences

$$g\preceq' e_G \text{ if and only if } \phi_{\preceq'}(g).0>0 \text{ if and only if } \phi_\preceq(g).f(0)>f(0),$$

showing that $\preceq'$ is the preorder induced from the $\phi_\preceq(G)$-orbit of $f(0)$. Thus there are countably many possibilities for $\preceq'$. ☐

We may now conclude the proof of Theorem N, which we restate for convenience.

Corollary 6.7.6 (Theorem N)**.** *Let G be a countable group. The semiconjugacy relation between cocompact actions of G on the line is essentially countable, and it is smooth if and only if $G\in\mathcal{C}$.*

Proof. Semiconjugacy on $\mathrm{Rep}_{\mathrm{cc}}(G)$ is Borel reducible to $\mathcal{R}$ by the first statement of Proposition 6.7.4, and both are essentially countable by Proposition 6.7.5.

Conjugacy on $\mathrm{Rep}_{\mathrm{min}}(G)\cup\mathrm{Rep}_{\mathrm{cyc}}(G)$ is clearly Borel reducible to semiconjugacy on $\mathrm{Rep}_{\mathrm{cc}}(G)$, and Proposition 6.7.4 shows that both relations are bireducible to each other. The proof of Proposition 6.7.5 shows that conjugacy on $\mathrm{Rep}_{\mathrm{cyc}}(G)$ is always smooth, so semiconjugacy on $\mathrm{Rep}_{\mathrm{cc}}(G)$ is smooth if and only if $G\in\mathcal{C}$. ☐

Remark. Recall that two cocompact actions $\varphi_1,\varphi_2\in\mathrm{Rep}_{\mathrm{cc}}(G)$ are *pointed semiconjugate* if there is an action $\eta\in\mathrm{Rep}_{\mathrm{cc}}(G)$ that is minimal or cyclic, and semiconjugacies $h_i\colon\mathbb{R}\to\mathbb{R}$ between φ_i and η for $i=1,2$ such that $h_1(0)=h_2(0)$. It is not hard to see that the map η_1 from the proof of Proposition 6.7.4 exhibits a Borel reduction of pointed semiconjugacy on $\mathrm{Rep}_{\mathrm{cc}}(G)$ to equality on $\mathrm{LPO}_{\mathrm{can}}(G)$. Therefore the relation of pointed semiconjugacy is always smooth.

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Aspects dynamiques et probabilistes des actions proximales de groupe

Résumé abrégé : Cette thèse est consacrée à l'étude des actions proximales de groupes dénombrables en dimension un et zéro depuis la perspective de la dynamique et des probabilités. Les résultats sont de nature assez variée. Un fil conducteur, parmi d'autres, est la présence des groupes de Thompson F, T et V ; presque la totalité des résultats s'appliquent à l'un de ces exemples, et sont parfois motivés par ceux-ci.

Mots clés : groupes dénombrables, dynamique unidimensionnelle, marches aléatoires sur des groupes, dynamique stationnaire.

Dynamical and probabilistic aspects of proximal group actions

Brief abstract : This thesis is devoted to the study of proximal actions of countable groups in dimensions one and zero from the perspective of dynamics and probability. The character of the results is fairly diverse. A common theme, among others, is the presence of Thompson's groups F, T and V ; almost all the results can be applied to one of these examples, and are sometimes motivated by them.

Keywords : countable groups, one-dimensional dynamics, random walks on groups, stationary dynamics.

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